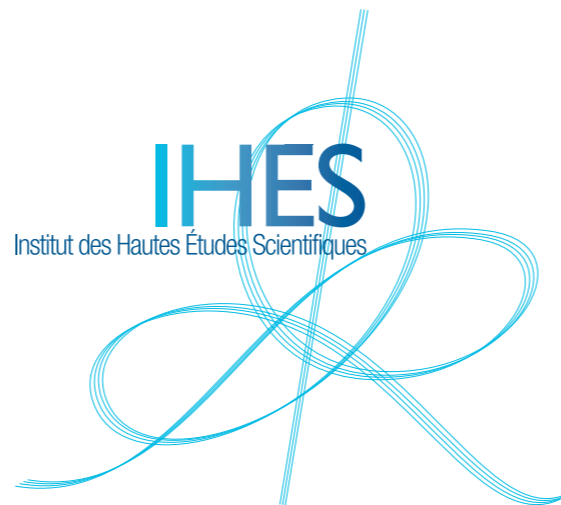


Motion of Binary Black Holes: using Quantum ideas to advance Classical dynamics

Thibault Damour
Institut des Hautes Etudes Scientifiques

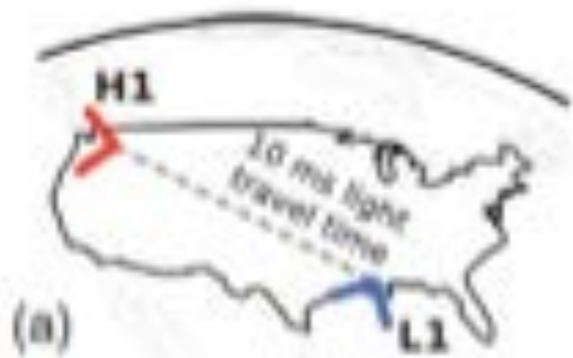


Conference on 90 Years of Quantum Mechanics
Institute of Advanced Studies at
Nanyang Technological University
23-26 January 2017, Singapore

THE TWO LIGO INTERFEROMETRIC GRAVITATIONAL WAVE DETECTORS



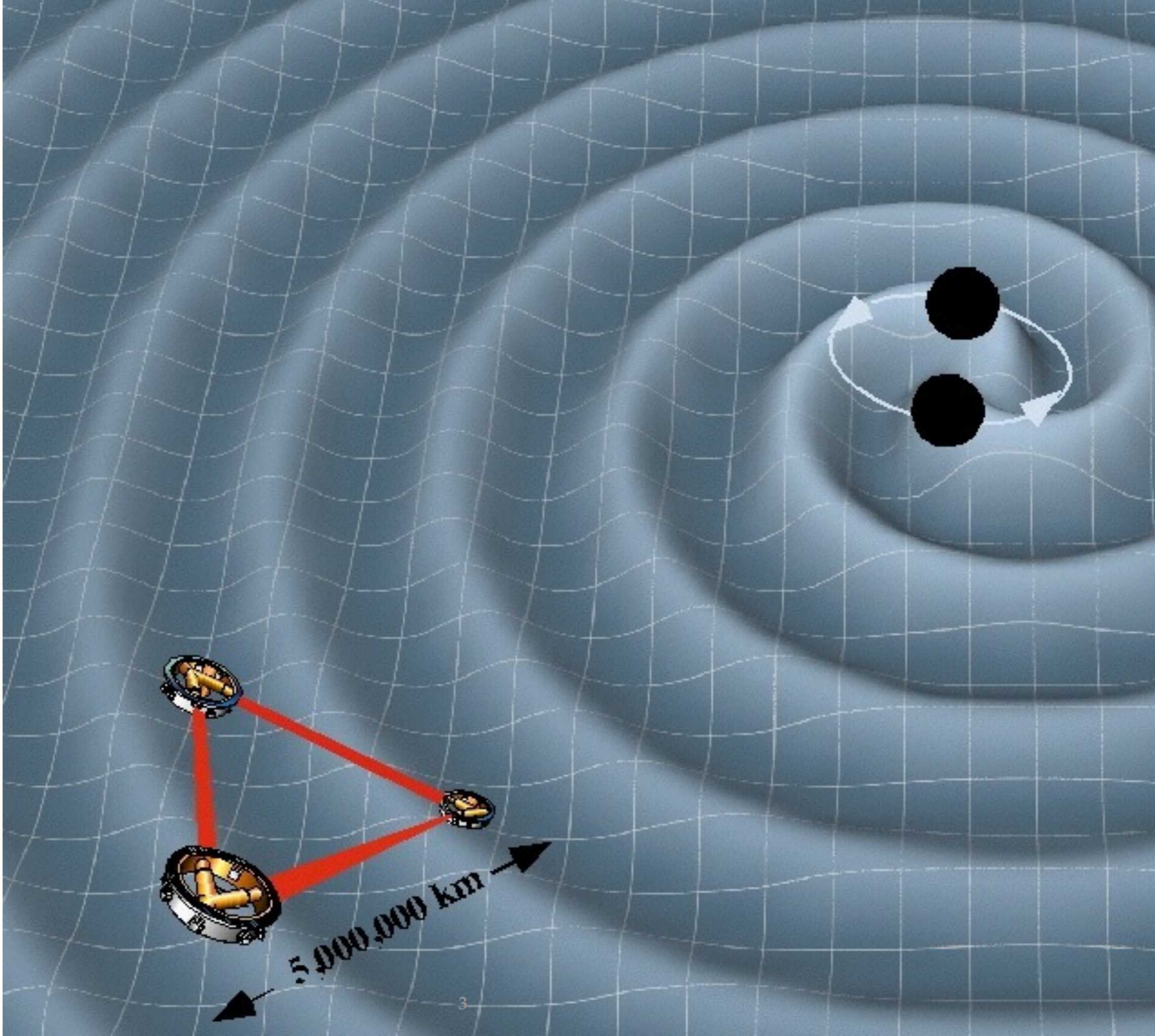
LIGO Hanford



LIGO Livingston



$$m_1 = 36^{+5}_{-4} M_{\odot}$$
$$m_2 = 29^{+4}_{-4} M_{\odot}$$
$$\chi_{\text{eff}} = -0.06^{+0.17}_{-0.18}$$
$$D_L = 410^{+160}_{-180} \text{Mpc}$$



LASER INTERFEROMETER GW DETECTORS

[Weber]

1962 Gertsenshtein-Pustovoi

1970 Forward

1972 Weiss

1972 Levine-Hall, Levine-Stebbins

1979 Billing H, Maischberger K,
Ruediger A, Schilling R, Schnupp L,
Winkler, W;

1979 Drever, Hough, Pugh, Edelstein,
Ward, Ford, Robertson

1981 Drever et al

1982 Brillet, Man, Vinet

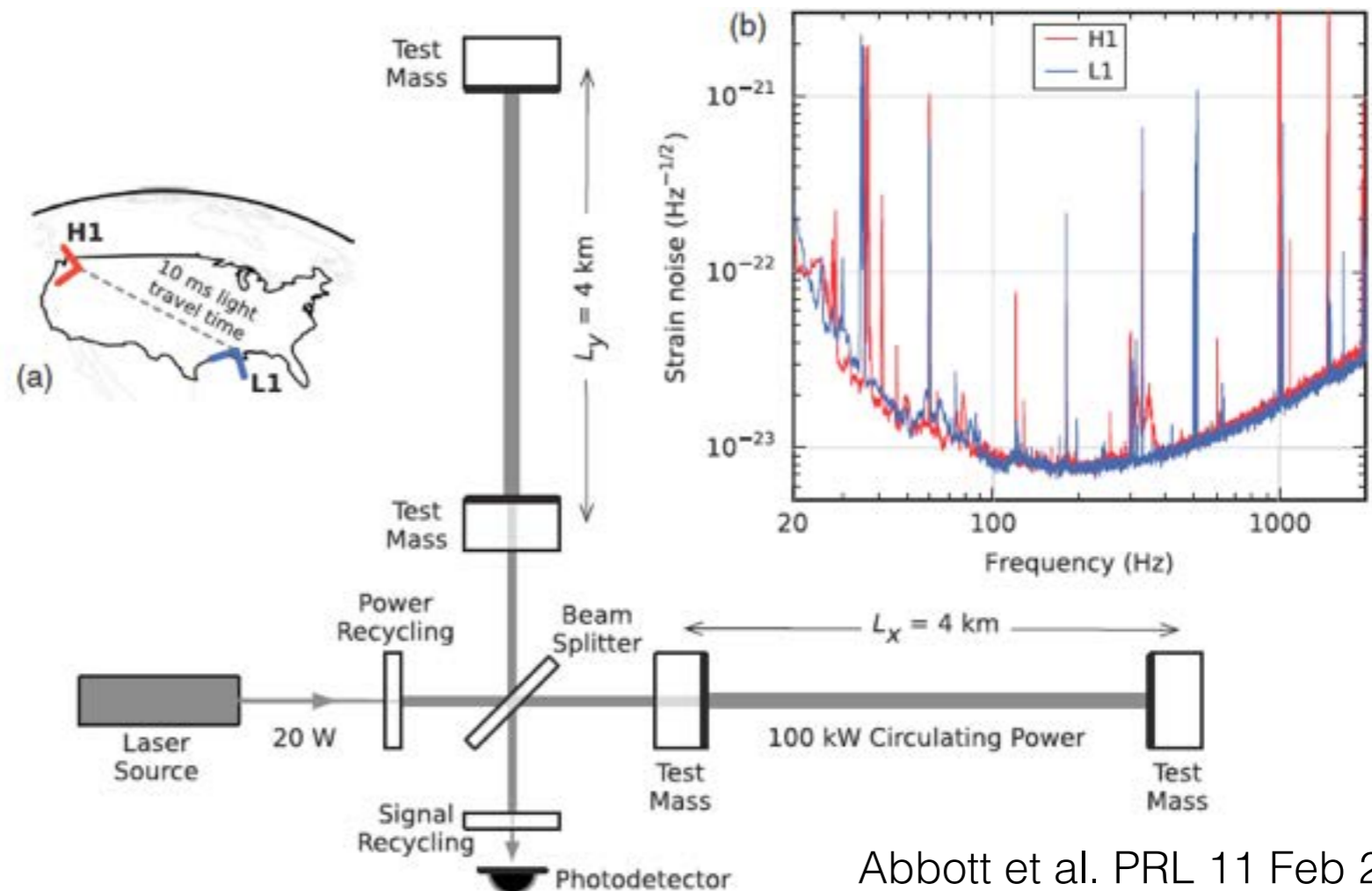
1980 Giazzoto

1988 Shoemaker et al

1988 Meers

.....

2016 Abbott et al. LIGO-Virgo collab.



Abbott et al. PRL 11 Feb 2016



LIGO Hanford



LIGO Livingston

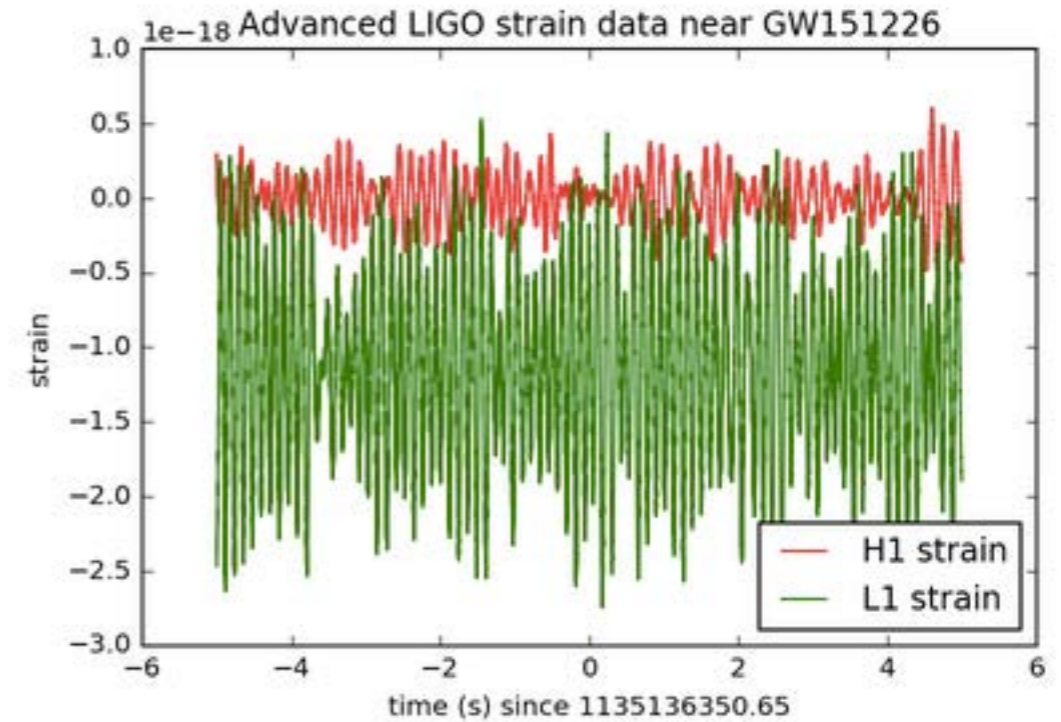
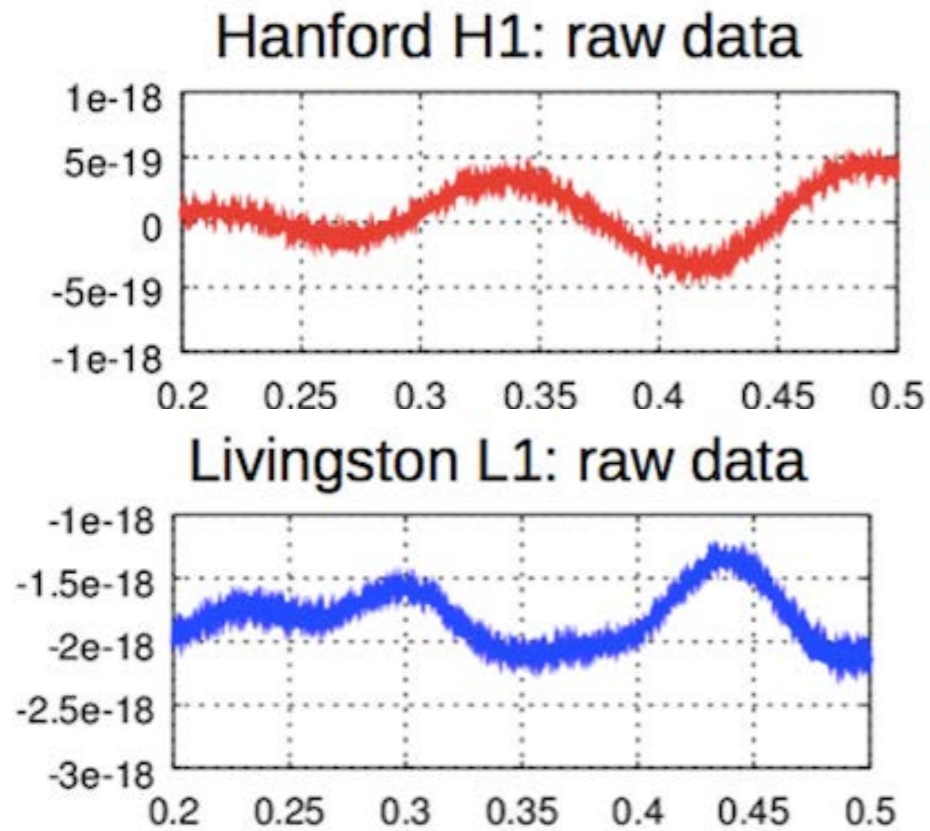


Virgo (IT)

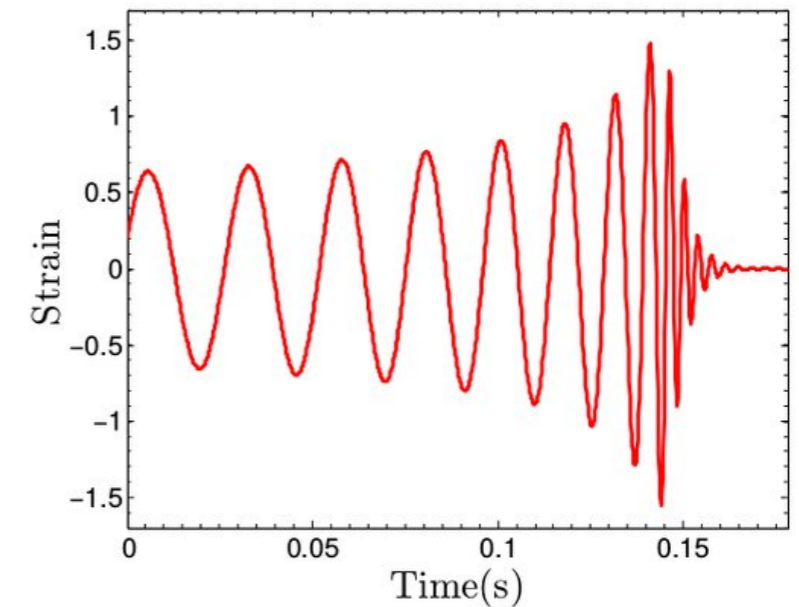
GW150914 and GW151226: incredibly small signals lost in the broad-band noise

GW150914, from Chassande-Mottin 5 April 2016

GW151226 from LIGO open data



$$h_{GW}^{\max} \sim 10^{-21} \sim 10^{-3} h_{LIGO}^{\text{broadband}}$$



Matched Filtering

$$\langle output | h_{\text{template}} \rangle = \int \frac{df}{S_n(f)} o(f) h_{\text{template}}^*(f)$$

MATCHED FILTERING SEARCH AND DATA ANALYSIS

Precomputed bank of 250 000 templates for inspiralling and coalescing BBH GW waveforms: m_1 , m_2 , $\chi_1 = S_1/m_1^2$, $\chi_2 = S_2/m_2^2$

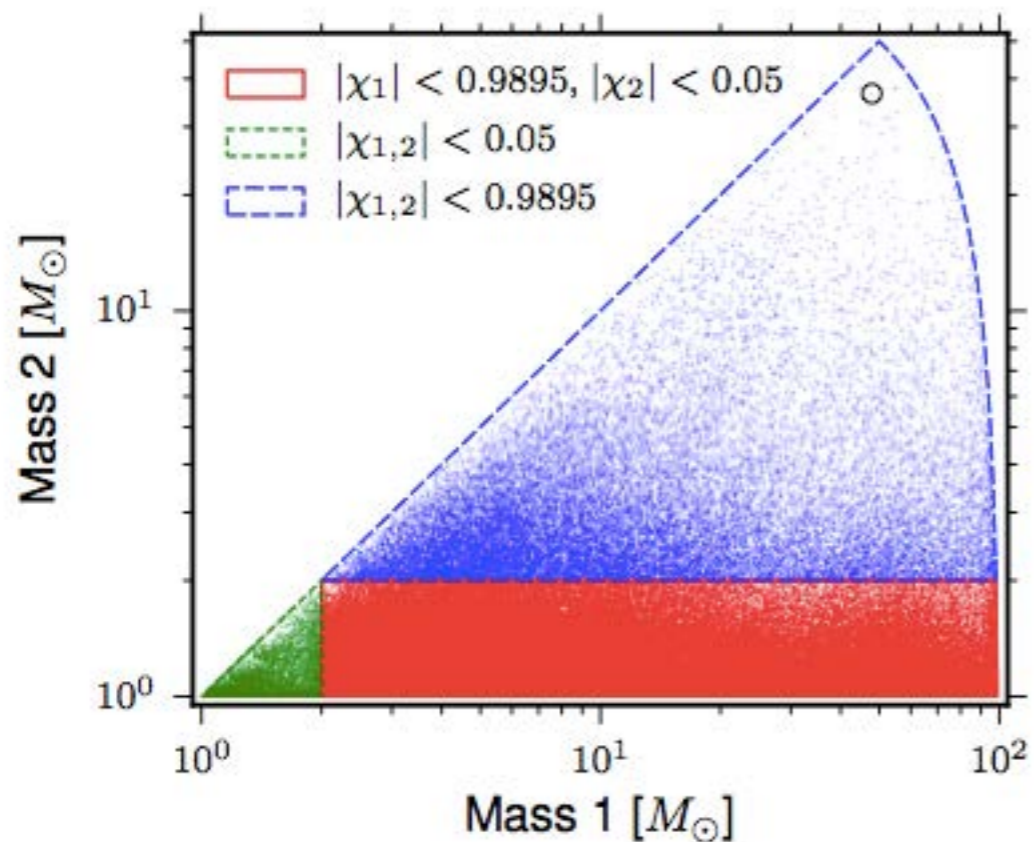
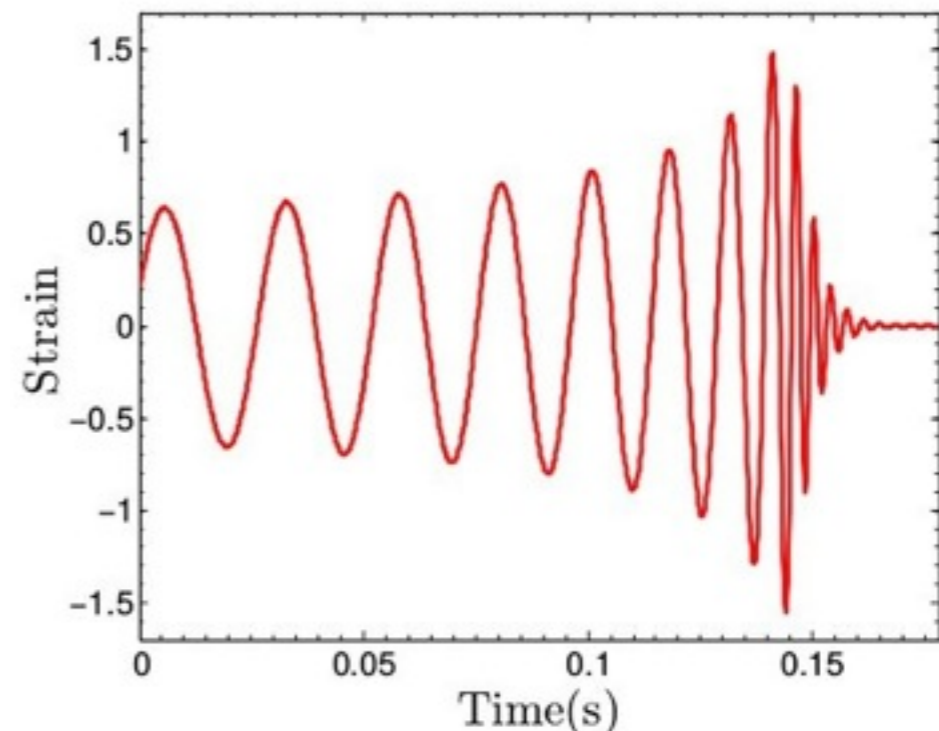


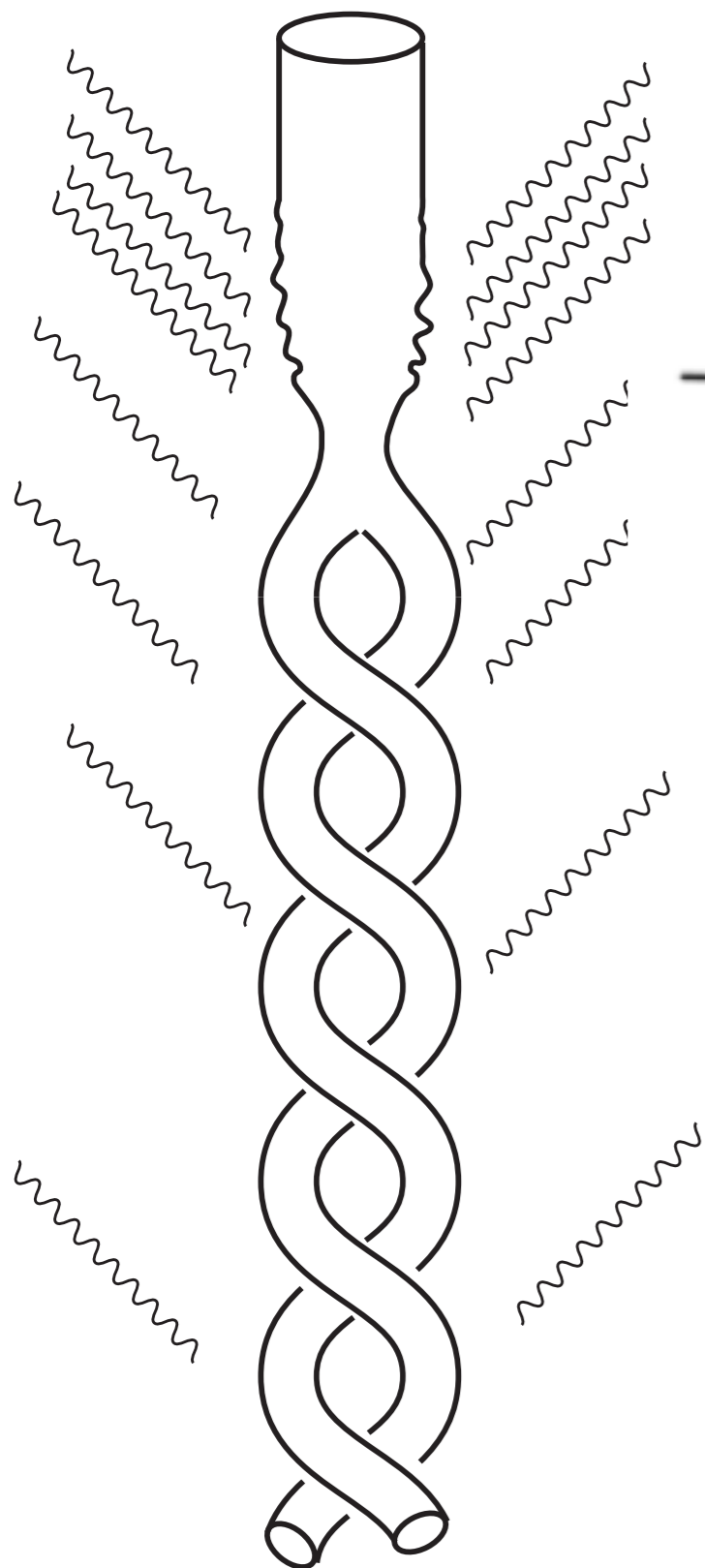
FIG. 1. The four-dimensional search parameter space covered by the template bank shown projected into the component-mass plane, using the convention $m_1 > m_2$. The lines bound mass regions with different limits on the dimensionless aligned-spin parameters χ_1 and χ_2 . Each point indicates the position of a template in the bank. The circle highlights the template that best matches GW150914. This does not coincide with the best-fit parameters due to the discrete nature of the template bank.

Search template bank made of **analytical EOB[NR] templates** (Buonanno-Damour 99..., Taracchini et al. 14) in ROM form; [post-computed NR waveform for GW151226 took three months and 70 000 CPU hours !] Several data analysis pipelines: PyCBC, GstLAL (+ Phenom[EOB+NR])



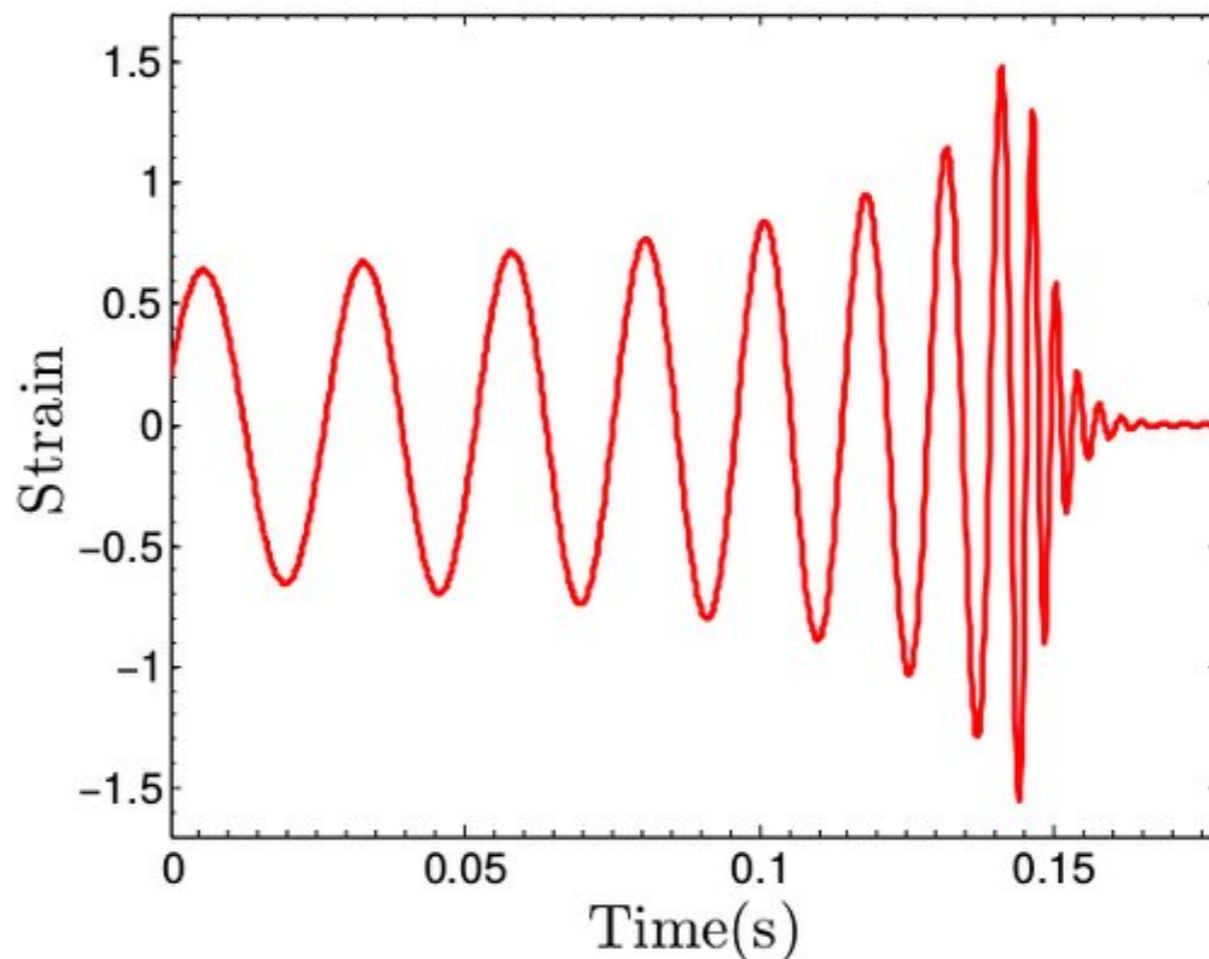
$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \quad R_{\mu\nu} = 0$$

$$ds^2 = g_{\mu\nu}(x^\lambda) dx^\mu dx^\nu$$



$$-g^{\mu\nu} g_{\alpha\beta, \mu\nu} + g^{\mu\nu} g^{\rho\sigma} (g_{\alpha\mu, \rho} g_{\beta\nu, \sigma} - g_{\alpha\mu, \rho} g_{\beta\sigma, \nu} + g_{\alpha\mu, \rho} g_{\nu\sigma, \beta} + g_{\beta\mu, \rho} g_{\nu\sigma, \alpha} - \frac{1}{2} g_{\mu\rho, \alpha} g_{\nu\sigma, \beta})$$

= 0

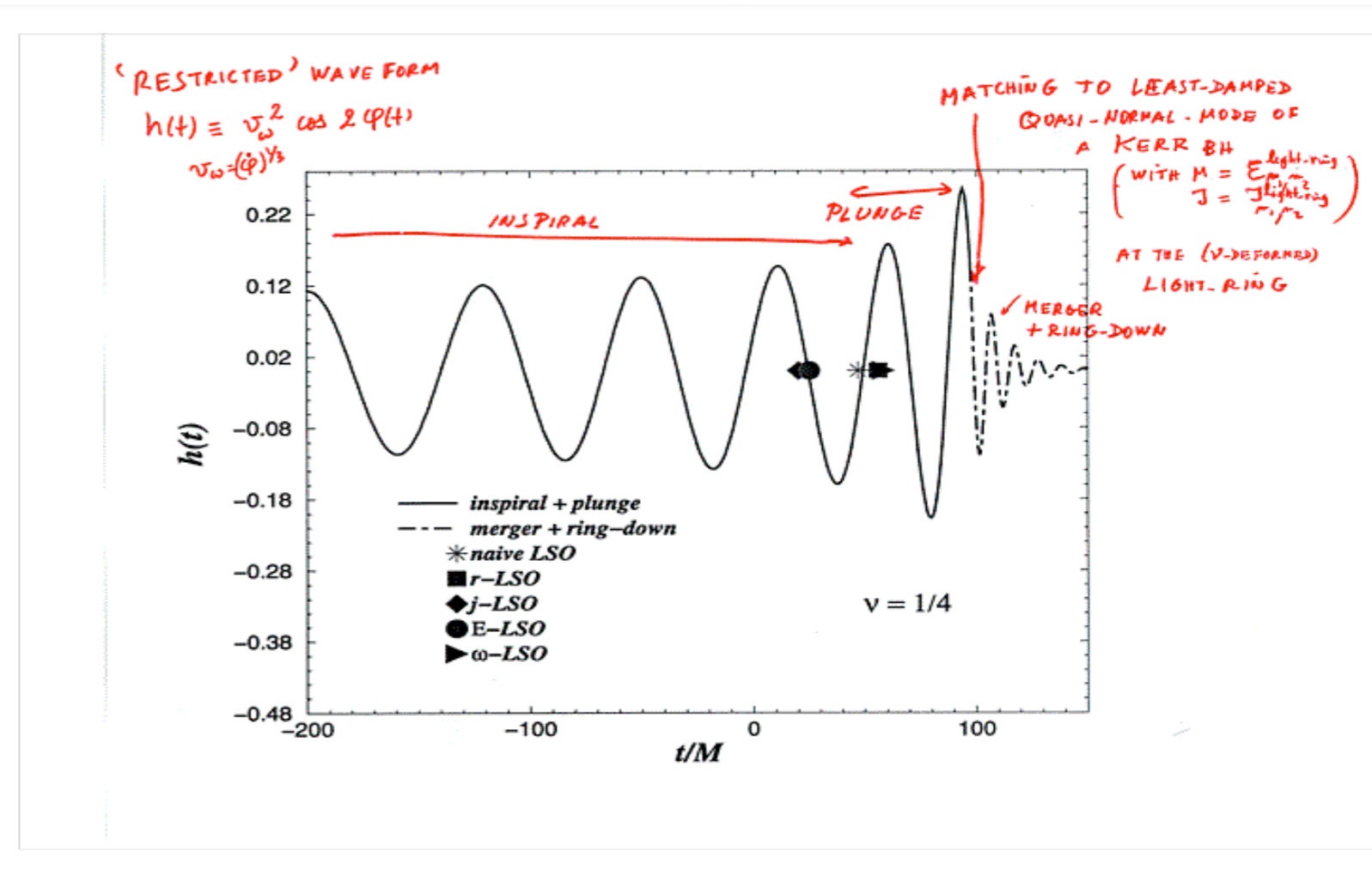
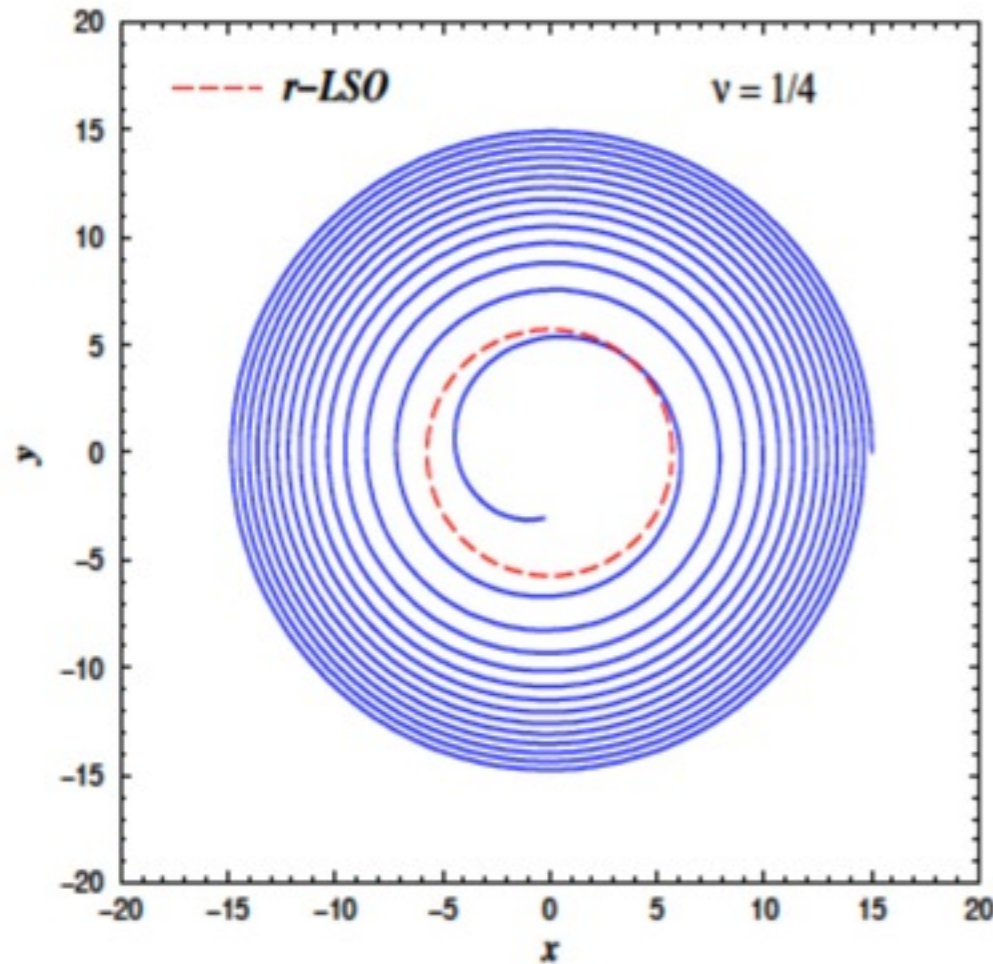


Effective One Body (EOB) Method

Buonanno-Damour 1999, 2000; Damour-Jaranowski-Schaefer 2000; Damour 2001

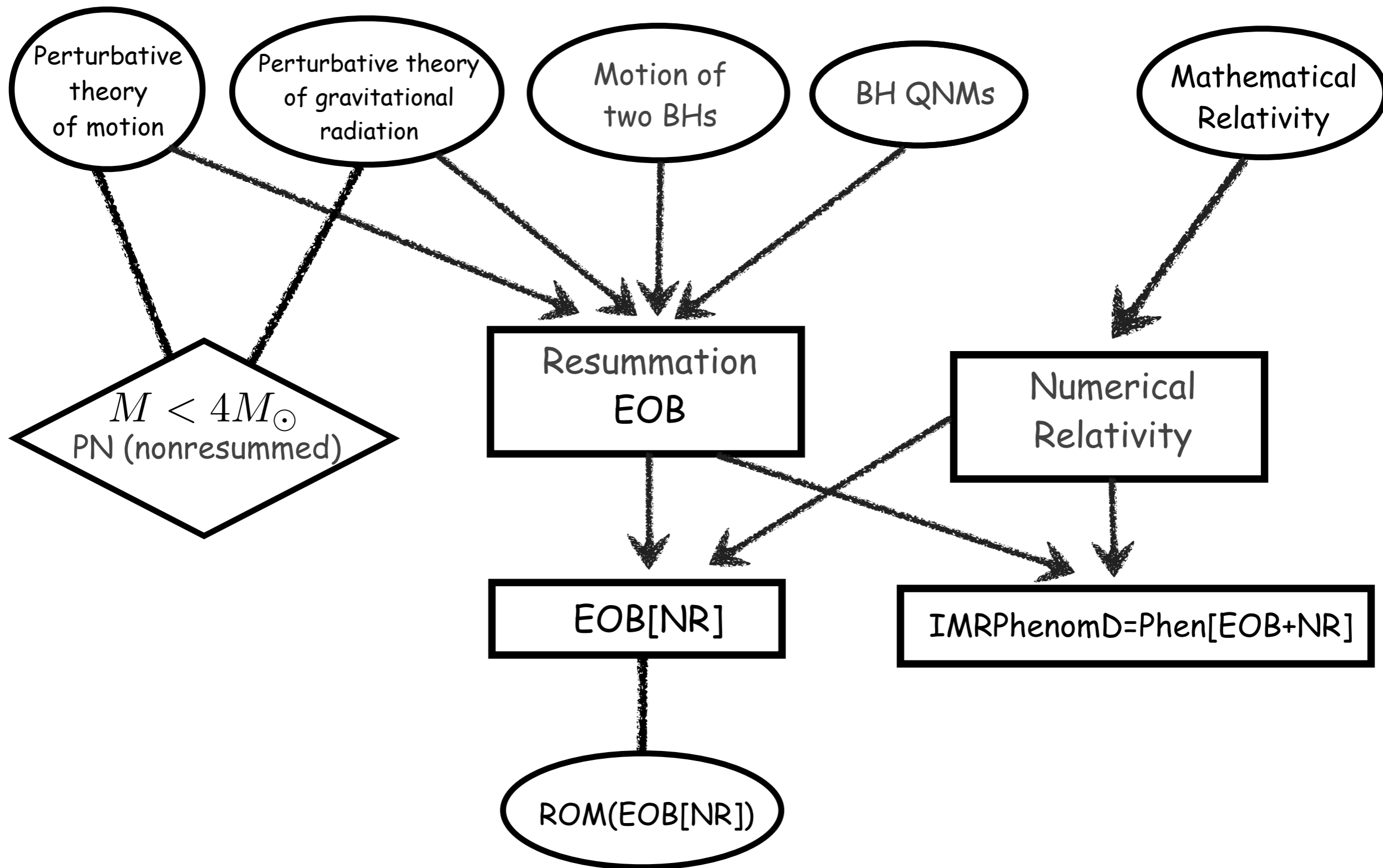
Resummation of perturbative PN results \longrightarrow description of the coalescence
 + addition of ringdown (Vishveshwara 70, Davis-Ruffini-Tiomno 1972)

Buonanno-Damour 2000



Predictions as early as 2000 :

continued transition, non adiabaticity, first complete waveform, final spin (OK within 10%), final mass



USING QUANTUM IDEAS TO ADVANCE CLASSICAL DYNAMICS

Diagrammatic techniques in gravity computations

Regularization and renormalization for dealing with
(classical) point masses

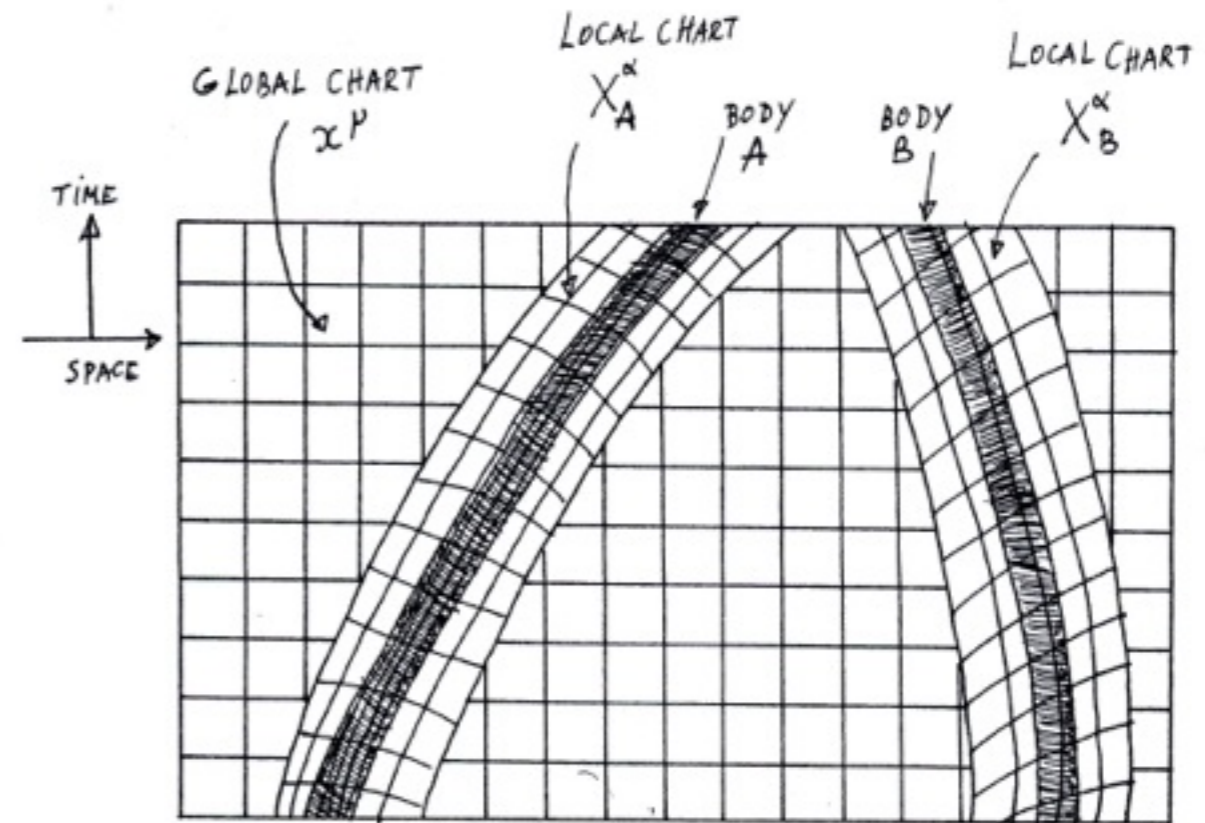
Use of quantum scattering amplitudes in classical gravity

A quantum-inspired correspondence in
binary Black Hole dynamics: EOB formalism

One-loop effects in a Black Hole background

Practical Techniques for Computing the Motion of Compact Bodies (NS or BH)

Multi-chart approach to motion of strong-self-gravity bodies, and **matched asymptotic expansions**
Manasse '63, Demianski-Grishchuk '74, D'Eath '75, Kates '80, Damour '82



Skeletonization : $T_{\mu\nu} \rightarrow$ point-masses (Mathisson '31)

delta-functions in GR : Infeld '54, Infeld-Plebanski '60

justified by Matched Asymptotic Expansions

(« **Effacing Property** » Damour '83)

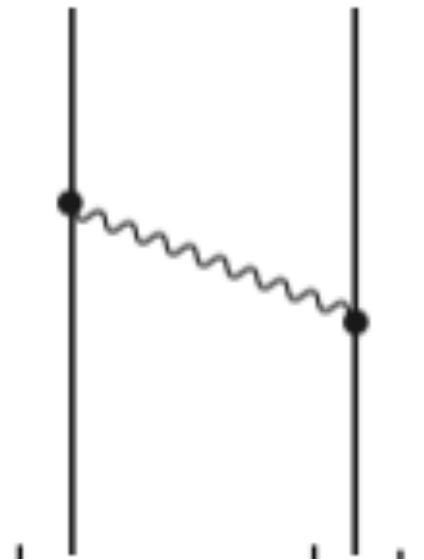
Fokker Action in Electrodynamics (1929)

$$S_{\text{tot}}[x_a^\mu, A_\mu] = - \sum_a \int m_a ds_a + \sum_a \int e_a dx_a^\mu A_\mu(x_a) - \int d^D x \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}$$

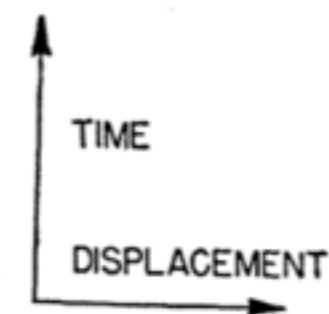
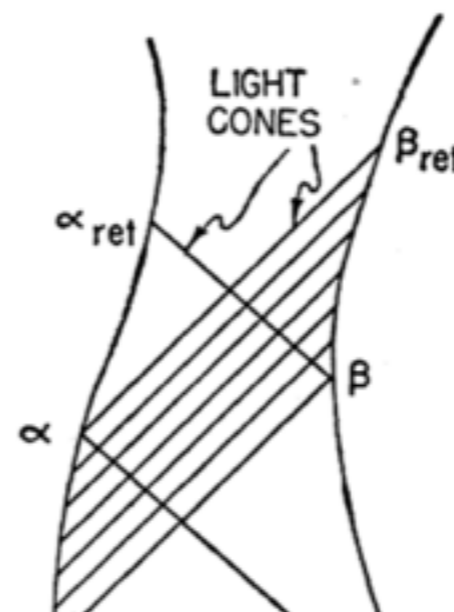
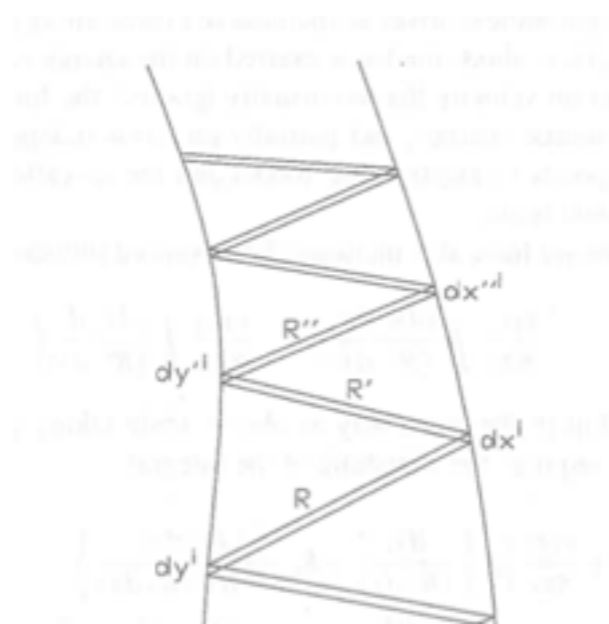
« Integrate out » the field A_μ in the total (particle+field) action

$$S_{\text{eff}}^{\text{class}}[x_a(s_a)] = - \sum_a m_a \int ds_a + \frac{1}{2} \sum_{a,b} e_a e_b \iint dx_a^\mu dx_{b\mu} \delta((x_a - x_b)^2).$$

One-photon-exchange diagram



The effective action $S_{\text{eff}}(x_a)$ was heavily used in the (second) Wheeler-Feynman paper (1949) together with similar diagrams to those used by Fokker



Fokker-type Action in Gravity and its Diagrammatic Expansion

Damour-Esposito-Farese '96

$$S_{\text{eff}}^{\text{class}}[x_a(s_a)] = [S_{\text{pm}} + S_{\text{EH}} + S_{\text{gf}}]_{g_{\mu\nu}(x) \rightarrow g_{\mu\nu}^{\text{gf}}[x_a(s_a)]} :$$

Needs gauge-fixed* action and time-symmetric Green function G .

*E.g. Arnowitt-Deser-Misner Hamiltonian formalism or harmonic coordinates.

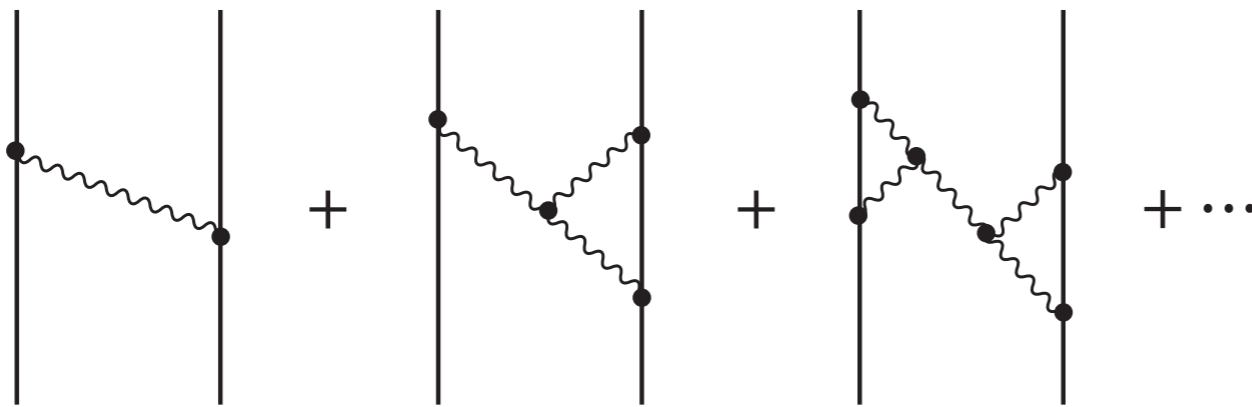
Perturbatively solving (in dimension $D=4 - \epsilon$) Einstein's equations to get the equations of motion and the action for the conservative dynamics

$$g = \eta + h$$

$$S(h, T) = \int \left(\frac{1}{2} h \square h + \partial \partial h h h + \dots + (h + h h + \dots) T \right)$$

$$\square h = -T + \dots \rightarrow h = G T + \dots$$

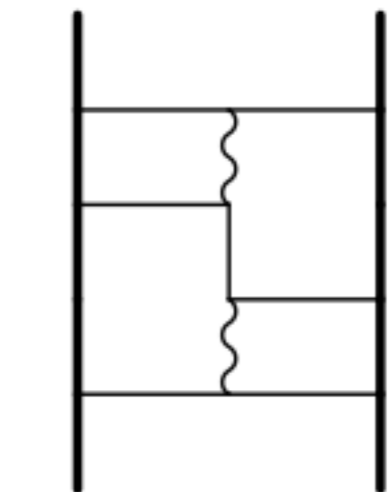
$$S_{\text{red}}(T) = \frac{1}{2} T G T + V_3(G T, G T, G T) + \dots$$



$O(G)$ = Newtonian
+ $(v/c)^n$ corrections

$O(G^2)$ = 1PN
= 1 loop

$O(G^3)$ = 2PN
= 2 loop



$O(G^5)$ = 4PN
= 4 loop

PN computation of the Fokker-type Gravity Action

PN expansion: in powers of $1/c^2$: $1\text{PN} = (v/c)^2$; $2\text{PN} = (v/c)^4$, etc $n\text{PN} = (v/c)^{2n}$

$$\square^{-1} = \left(\Delta - \frac{1}{c^2} \partial_t^2\right)^{-1} = \Delta^{-1} + \frac{1}{c^2} \partial_t^2 \Delta^{-2} + \dots$$

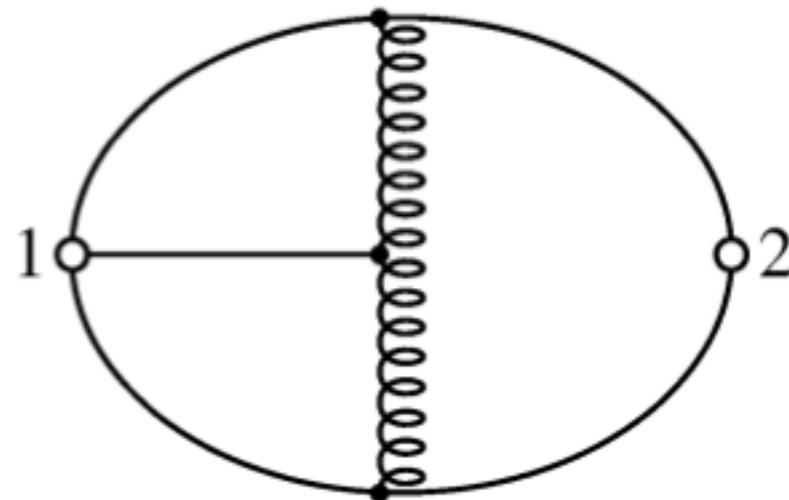
Beyond 1-loop order needs to use PN-expanded Green function for explicit computations.

This transforms spacetime diagrams (between two worldlines) into
(massless) two-point space diagrams (in three dimensions)

E.g. at 3PN, a **3-loop space diagram**

$$\sim G^4 m_1^3 m_2^2$$

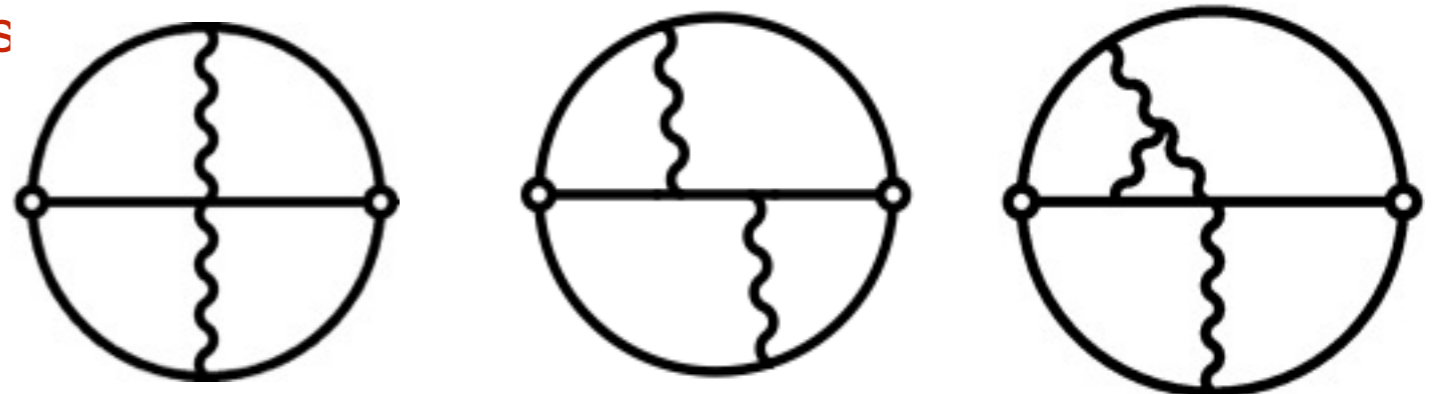
(Damour-Jaranowski-Schaefer 2001)



E.g. at 4PN, some **4-loop space diagrams**

$$\sim G^5 m_1^3 m_2^3$$

(Damour-Jaranowski 2017)



Regularization and Renormalization

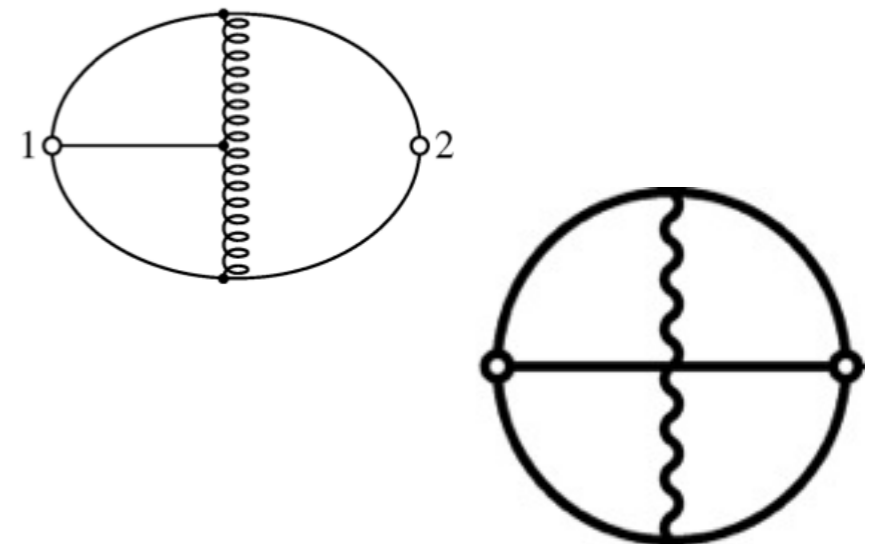
Like in Dirac 1938, the use of point-particles (delta functions) introduces **UV** divergences (linked to **self-field effects**).

It has been shown (« **Effacing property** » Damour '83) that possible internal-structure dependence in strong self-gravity objects (NSs, BHs) only arise at 5PN= 5-loop level

Below 5PN (5-loop) point-masses in GR are renormalizable
(Damour '83, Goldberger-Rothstein '06)

One still needs a regularization method that respects gravity's gauge invariance:
dimensional regularization $D=4+\epsilon$ ('t Hooft-Veltman '72)

It has been explicitly shown that S_{eff} was **UV finite** (in ADM gauge, and in dim.reg.) at
3 loops (Damour-Jaranowski-Schäfer '01) and
4 loops (Damour-Jaranowski-Schäfer '14, Jaranowski-Schäfer '15)



There appear **IR divergences** at 4PN (4 loop) linked to non-locality (Blanchet-Damour '88).

Alternative Computation of Effective Action

Instead of classically « integrating out » the field dofs

$$S_{\text{eff}}^{\text{class}}[x_a(s_a)] = [S_{\text{pm}} + S_{\text{EH}} + S_{\text{gf}}]_{g_{\mu\nu}(x) \rightarrow g_{\mu\nu}^{\text{gf}}[x_a(s_a)]}$$

Formal functional integral over the field (QED: Feynman '50; ...;

GR: « **Effective Field Theory** » approach

(Goldberger-Rothstein '06, Porto '06, Gilmore-Ross' 08, Levi '10, Foffa-Sturani '11 '13, Levi-Steinhoff '14, '15; Foffa-Mastrolia-Sturani-Sturm'16, Damour-Jaranowski '17)

$$e^{\frac{i}{\hbar} S_{\text{eff}}^{\text{quant}}} = \int Dg_{\mu\nu} e^{\frac{i}{\hbar} (S_{\text{pm}} + S_{\text{EH}} + S_{\text{gf}})}$$

Saddle-point estimation: $S_{\text{eff}}^{\text{quant}}[x_a(s_a)] = S_{\text{eff}}^{\text{class}}[x_a(s_a)] + O(\hbar)$.

However, the explicit computations are

done differently:

by means of

Wick's theorem,

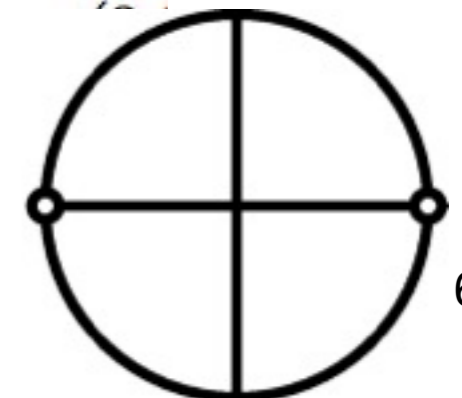
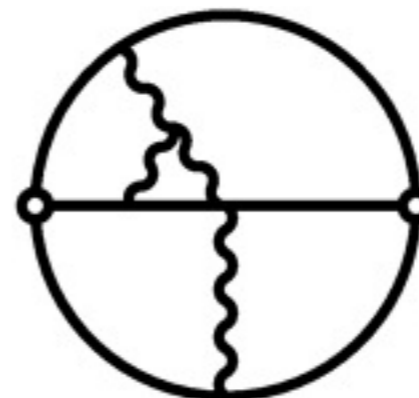
and p-space integrations

$$e^{\frac{i}{\hbar} S_{\text{eff}}} = \int D\varphi e^{\frac{i}{\hbar} (\int [\frac{1}{2} \varphi \mathcal{K} \varphi + \varphi s + g \varphi^3 + \dots])}$$

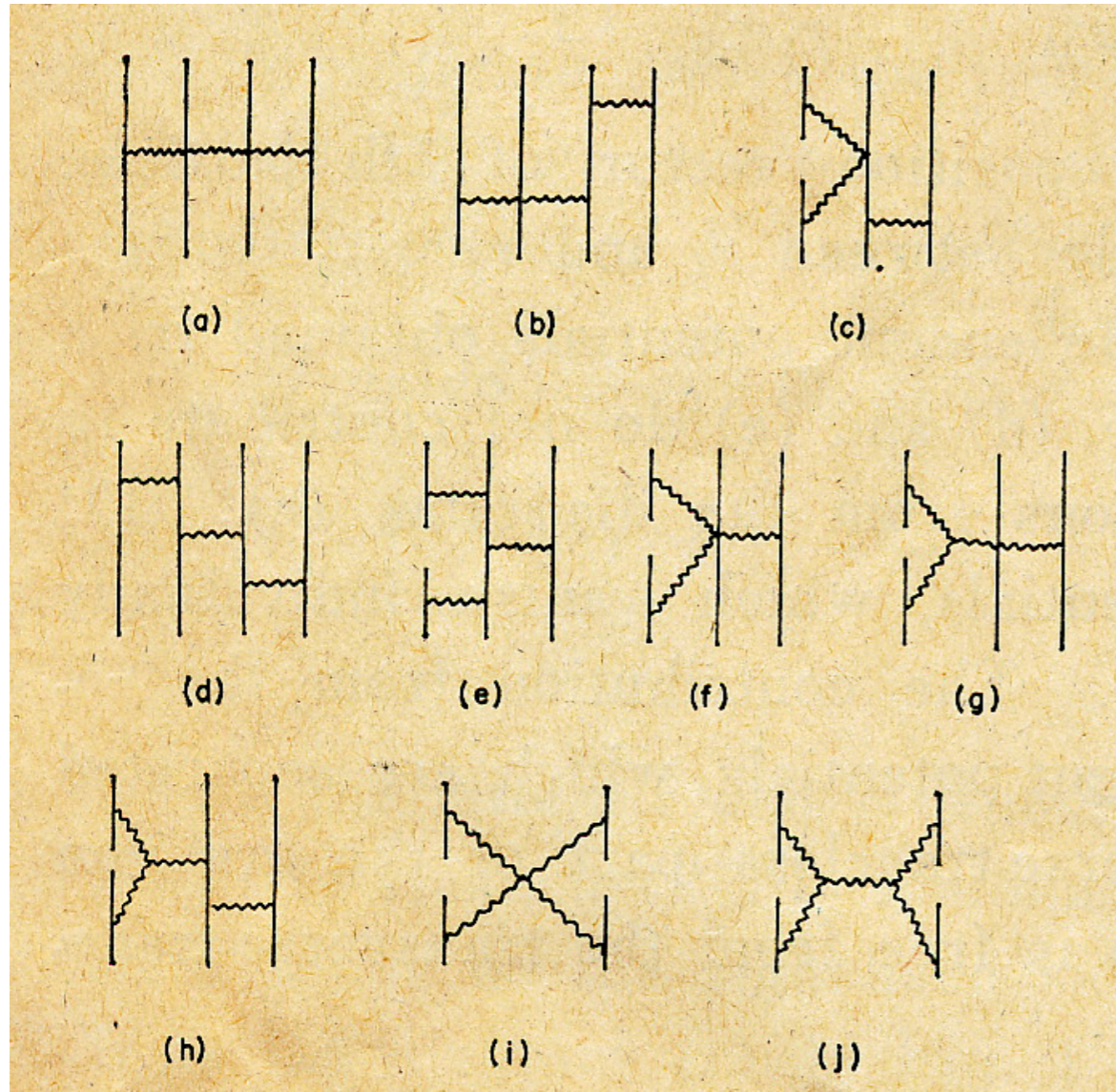
$$\int D\varphi e^{\frac{i}{\hbar} \int [\frac{1}{2} \varphi \mathcal{K} \varphi]} \sum \frac{(i/\hbar)^n}{n!} \left(\int (\varphi s + g \varphi^3 + \dots) \right)^n$$

$$\langle \varphi(x) \varphi(y) \rangle = \int D\varphi e^{\frac{i}{\hbar} \int [\frac{1}{2} \varphi \mathcal{K} \varphi]} \varphi(x) \varphi(y) = i \hbar \mathcal{K}_{x,y}^{-1}$$

Recently (Damour-Jaranowski '17) found errors in the EFT computation (by Foffa-Mastrolia-Sturani-Sturm'16) of some of the static 4-loop contributions, and found a way of analytically computing an "uncomputable" 2-point 4-loop integral



Another Quantum-based Technique



- Quantum Scattering Amplitudes \longrightarrow Potential

one-graviton exchange :

Corinaldesi '56 '71,

Barker-Gupta-Haracz 66,

Barker-O'Connell 70, Hiida-Okamura 72

Nonlinear: Iwasaki 7 [1PN],

Okamura-Ohta-Kimura-Hiida 73[2 PN]

Recent progress in gravity amplitudes (double-copy Bern et al.) could be useful to improve the classical scattering (Damour '16; Bjerrum-Bohr—Vanhove, in preparation)

Post-Newtonian Equations of Motion [2-body, wo spins]

- 1PN (including v^2/c^2) [Lorentz-Droste '17], Einstein-Infeld-Hoffmann '38
- 2PN (inc. v^4/c^4) Ohta-Okamura-Kimura-Hiida '74, Damour-Deruelle '81
Damour '82, Schäfer '85, Kopeikin '85
- 2.5 PN (inc. v^5/c^5) Damour-Deruelle '81, Damour '82, Schäfer '85,
Kopeikin '85
- 3 PN (inc. v^6/c^6) Jaranowski-Schäfer '98, Blanchet-Faye '00,
Damour-Jaranowski-Schäfer '01, Itoh-Futamase '03,
Blanchet-Damour-Esposito-Farèse' 04, Foffa-Sturani '11
- 3.5 PN (inc. v^7/c^7) Iyer-Will '93, Jaranowski-Schäfer '97, Pati-Will '02,
Königsdörffer-Faye-Schäfer '03, Nissanke-Blanchet '05, Itoh '09
- **4PN** (inc. v^8/c^8) Jaranowski-Schäfer '13, Foffa-Sturani '13,'16
Bini-Damour '13, Damour-Jaranowski-Schäfer '14, Bernard et al'16

New feature : **non-locality in time**

2-body Taylor-expanded N + 1PN + 2PN Hamiltonian

$$H_N(\mathbf{x}_a, \mathbf{p}_a) = \frac{\mathbf{p}_1^2}{2m_1} - \frac{1}{2} \frac{Gm_1m_2}{r_{12}} + (1 \leftrightarrow 2)$$

$$c^2 H_{1PN}(\mathbf{x}_a, \mathbf{p}_a) = -\frac{1}{8} \frac{(\mathbf{p}_1^2)^2}{m_1^3} + \frac{1}{8} \frac{Gm_1m_2}{r_{12}} \left(-12 \frac{\mathbf{p}_1^2}{m_1^2} + 14 \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1m_2} + 2 \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1m_2} \right) \\ + \frac{1}{4} \frac{Gm_1m_2}{r_{12}} \frac{G(m_1 + m_2)}{r_{12}} + (1 \leftrightarrow 2),$$

$$c^4 H_{2PN}(\mathbf{x}_a, \mathbf{p}_a) = \frac{1}{16} \frac{(\mathbf{p}_1^2)^3}{m_1^5} + \frac{1}{8} \frac{Gm_1m_2}{r_{12}} \left(5 \frac{(\mathbf{p}_1^2)^2}{m_1^4} - \frac{11}{2} \frac{\mathbf{p}_1^2 \mathbf{p}_2^2}{m_1^2 m_2^2} - \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} + 5 \frac{\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} \right. \\ \left. - 6 \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^2 m_2^2} - \frac{3}{2} \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} \right) \\ + \frac{1}{4} \frac{G^2 m_1 m_2}{r_{12}^2} \left(m_2 \left(10 \frac{\mathbf{p}_1^2}{m_1^2} + 19 \frac{\mathbf{p}_2^2}{m_2^2} \right) - \frac{1}{2} (m_1 + m_2) \frac{27(\mathbf{p}_1 \cdot \mathbf{p}_2) + 6(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1 m_2} \right) \\ - \frac{1}{8} \frac{Gm_1m_2}{r_{12}} \frac{G^2(m_1^2 + 5m_1m_2 + m_2^2)}{r_{12}^2} + (1 \leftrightarrow 2),$$

2-body Taylor-expanded 3PN Hamiltonian [JS 98, DJS 01]

$$\begin{aligned}
 c^6 H_{3\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) = & -\frac{5}{128} \frac{(\mathbf{p}_1^2)^4}{m_1^7} + \frac{1}{32} \frac{Gm_1 m_2}{r_{12}} \left(-14 \frac{(\mathbf{p}_1^2)^3}{m_1^6} + 4 \frac{((\mathbf{p}_1 \cdot \mathbf{p}_2)^2 + 4\mathbf{p}_1^2 \mathbf{p}_2^2) \mathbf{p}_1^2}{m_1^4 m_2^2} + 6 \frac{\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^4 m_2^2} \right. \\
 & - 10 \frac{(\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2 + \mathbf{p}_2^2 (\mathbf{n}_{12} \cdot \mathbf{p}_1)^2) \mathbf{p}_1^2}{m_1^4 m_2^2} + 24 \frac{\mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2) (\mathbf{n}_{12} \cdot \mathbf{p}_1) (\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^4 m_2^2} \\
 & + 2 \frac{\mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2) (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^3 m_2^3} + \frac{(7\mathbf{p}_1^2 \mathbf{p}_2^2 - 10(\mathbf{p}_1 \cdot \mathbf{p}_2)^2) (\mathbf{n}_{12} \cdot \mathbf{p}_1) (\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^3 m_2^3} \\
 & + \frac{(\mathbf{p}_1^2 \mathbf{p}_2^2 - 2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2) (\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1^3 m_2^3} + 15 \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2) (\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^3 m_2^3} \\
 & - 18 \frac{\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_1) (\mathbf{n}_{12} \cdot \mathbf{p}_2)^3}{m_1^3 m_2^3} + 5 \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^3}{m_1^3 m_2^3} \left. \right) + \frac{G^2 m_1 m_2}{r_{12}^2} \left(\frac{1}{16} (m_1 - 27m_2) \frac{(\mathbf{p}_1^2)^2}{m_1^4} \right. \\
 & - \frac{115}{16} m_1 \frac{\mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1^3 m_2} + \frac{1}{48} m_2 \frac{25(\mathbf{p}_1 \cdot \mathbf{p}_2)^2 + 371\mathbf{p}_1^2 \mathbf{p}_2^2}{m_1^2 m_2^2} + \frac{17}{16} \frac{\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_1)^2}{m_1^3} + \frac{5}{12} \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4}{m_1^3} \\
 & - \frac{1}{8} m_1 \frac{(15\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2) + 11(\mathbf{p}_1 \cdot \mathbf{p}_2) (\mathbf{n}_{12} \cdot \mathbf{p}_1)) (\mathbf{n}_{12} \cdot \mathbf{p}_1)}{m_1^3 m_2} - \frac{3}{2} m_1 \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3 (\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^3 m_2} \\
 & + \frac{125}{12} m_2 \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2) (\mathbf{n}_{12} \cdot \mathbf{p}_1) (\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^2 m_2^2} + \frac{10}{3} m_2 \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} \\
 & - \frac{1}{48} (220m_1 + 193m_2) \frac{\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} \left. \right) + \frac{G^3 m_1 m_2}{r_{12}^3} \left(-\frac{1}{48} \left(425m_1^2 + \left(473 - \frac{3}{4} \pi^2 \right) m_1 m_2 + 150m_2^2 \right) \frac{\mathbf{p}_1^2}{m_1^2} \right. \\
 & + \frac{1}{16} \left(77(m_1^2 + m_2^2) + \left(143 - \frac{1}{4} \pi^2 \right) m_1 m_2 \right) \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1 m_2} + \frac{1}{16} \left(20m_1^2 - \left(43 + \frac{3}{4} \pi^2 \right) m_1 m_2 \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2}{m_1^2} \\
 & + \frac{1}{16} \left(21(m_1^2 + m_2^2) + \left(119 + \frac{3}{4} \pi^2 \right) m_1 m_2 \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1) (\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1 m_2} \left. \right) \\
 & + \frac{1}{8} \frac{G^4 m_1 m_2^3}{r_{12}^4} \left(\left(\frac{227}{3} - \frac{21}{4} \pi^2 \right) m_1 + m_2 \right) + (1 \leftrightarrow 2).
 \end{aligned}$$

2-body Taylor-expanded 4PN Hamiltonian [DJS, 2014]

$$\begin{aligned}
 c^8 H_{4\text{PN}}^{\text{local}}(\mathbf{x}_a, \mathbf{p}_a) &= \frac{7(\mathbf{p}_1^2)^5}{256m_1^5} + \frac{Gm_1m_2}{r_{12}} H_{48}(\mathbf{x}_a, \mathbf{p}_a) + \frac{G^2m_1m_2}{r_{12}^2} m_1 H_{40}(\mathbf{x}_a, \mathbf{p}_a) \\
 &+ \frac{G^3m_1m_2}{r_{12}^3} (m_1^2 H_{441}(\mathbf{x}_a, \mathbf{p}_a) + m_1m_2 H_{442}(\mathbf{x}_a, \mathbf{p}_a)) \\
 &+ \frac{G^4m_1m_2}{r_{12}^4} (m_1^3 H_{421}(\mathbf{x}_a, \mathbf{p}_a) + m_1^2m_2 H_{422}(\mathbf{x}_a, \mathbf{p}_a)) \\
 &+ \frac{G^5m_1m_2}{r_{12}^5} H_{40}(\mathbf{x}_a, \mathbf{p}_a) + (1 \leftrightarrow 2), \tag{A3}
 \end{aligned}$$

$$\begin{aligned}
 H_{48}(\mathbf{x}_a, \mathbf{p}_a) &= \frac{45(\mathbf{p}_1^2)^4}{128m_1^4} - \frac{9(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{64m_1^2m_2^2} + \frac{15(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1^2)^3}{64m_1^2m_2^2} - \frac{9(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1^2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{16m_1^2m_2^2} \\
 &+ \frac{3(\mathbf{p}_1^2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{32m_1^2m_2^2} + \frac{15(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{64m_1^2m_2^2} + \frac{21(\mathbf{p}_1^2)^3\mathbf{p}_2^2}{64m_1^2m_2^2} + \frac{35(\mathbf{n}_{12} \cdot \mathbf{p}_1)^5(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3}{256m_1^5m_2^3} \\
 &+ \frac{25(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3\mathbf{p}_1^2}{128m_1^3m_2^3} + \frac{33(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3(\mathbf{p}_1^2)^2}{256m_1^3m_2^3} + \frac{85(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{256m_1^4m_2^2} \\
 &+ \frac{45(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{128m_1^2m_2^2} + \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1^2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{256m_1^2m_2^2} + \frac{25(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{64m_1^3m_2^2} \\
 &+ \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{64m_1^2m_2^2} + \frac{3(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^3}{64m_1^2m_2^2} + \frac{3\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^3}{64m_1^2m_2^2} + \frac{55(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2}{256m_1^3m_2^2} \\
 &+ \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{128m_1^3m_2^2} + \frac{25(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{256m_1^2m_2^2} + \frac{23(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{256m_1^4m_2^2} \\
 &+ \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{128m_1^2m_2^2} + \frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{256m_1^2m_2^2} + \frac{5(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^4\mathbf{p}_1^2}{64m_1^2m_2^4} + \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_2)^4(\mathbf{p}_1^2)^2}{64m_1^2m_2^4} \\
 &+ \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{4m_1^2m_2^3} + \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{16m_1^2m_2^3} + \frac{5(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3\mathbf{p}_2^2}{64m_1^4m_2^3} + \frac{21(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1^2)\mathbf{p}_2^2}{64m_1^2m_2^2} \\
 &+ \frac{3(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{32m_1^2m_2^2} + \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{4m_1^3m_2^2} + \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{16m_1^2m_2^2} + \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2\mathbf{p}_2^2}{16m_1^2m_2^2} \\
 &+ \frac{\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2\mathbf{p}_2^2}{32m_1^2m_2^2} + \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4(\mathbf{p}_1^2)^2}{64m_1^4m_2^2} + \frac{3(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{32m_1^3m_2^2} + \frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_2^2)^2}{128m_1^2m_2^2}. \tag{A4a}
 \end{aligned}$$

$$\begin{aligned}
 H_{40}(\mathbf{x}_a, \mathbf{p}_a) &= \frac{369(\mathbf{n}_{12} \cdot \mathbf{p}_1)^6}{160m_1^6} - \frac{889(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4\mathbf{p}_1^2}{192m_1^4} + \frac{49(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1^2)^2}{16m_1^2} - \frac{63(\mathbf{p}_1^2)^3}{64m_1^3} - \frac{549(\mathbf{n}_{12} \cdot \mathbf{p}_1)^5(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{128m_1^5m_2} \\
 &+ \frac{67(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2}{16m_1^3m_2} - \frac{167(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1^2)^2}{128m_1^2m_2} + \frac{1547(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4(\mathbf{p}_1 \cdot \mathbf{p}_2)}{256m_1^4m_2} - \frac{851(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{128m_1^3m_2} \\
 &+ \frac{1099(\mathbf{p}_1^2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{256m_1^2m_2} + \frac{3263(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{1280m_1^4m_2^2} + \frac{1067(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2}{480m_1^2m_2^2} + \frac{4567(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{3840m_1^2m_2^2} \\
 &+ \frac{3571(\mathbf{n}_{12} \cdot \mathbf{p}_1)^5(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{320m_1^5m_2} + \frac{3073(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{480m_1^3m_2} + \frac{4349(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{1280m_1^3m_2} \\
 &+ \frac{3461\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{3840m_1^2m_2^2} + \frac{1673(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4\mathbf{p}_1^2}{1920m_1^4m_2^2} - \frac{1999(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2\mathbf{p}_1^2\mathbf{p}_2^2}{3840m_1^2m_2^2} + \frac{2081(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{3840m_1^2m_2^2} - \frac{13(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3}{8m_1^3m_2^3} \\
 &+ \frac{191(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3\mathbf{p}_1^2}{192m_1^2m_2^3} - \frac{19(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{384m_1^2m_2^2} + \frac{5(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{384m_1^2m_2^2} \\
 &+ \frac{11(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{192m_1^2m_2^2} + \frac{77(\mathbf{p}_1 \cdot \mathbf{p}_2)^3}{96m_1^2m_2^2} + \frac{233(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_2^2}{96m_1^3m_2^2} - \frac{47(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{32m_1^2m_2^2} \\
 &+ \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{384m_1^2m_2^2} + \frac{185\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{384m_1^2m_2^2} - \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^4}{4m_1^2m_2^4} + \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_2)^4\mathbf{p}_1^2}{4m_1^2m_2^4} \\
 &+ \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3(\mathbf{p}_1 \cdot \mathbf{p}_2)}{2m_1^2m_2^3} + \frac{21(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{16m_1^2m_2^2} + \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_2^2}{6m_1^2m_2^2} + \frac{49(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{48m_1^2m_2^2} \\
 &+ \frac{133(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_1^2}{24m_1^2m_2^2} - \frac{77(\mathbf{p}_1 \cdot \mathbf{p}_2)^2\mathbf{p}_2^2}{96m_1^2m_2^2} + \frac{197(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1^2)^2}{96m_1^2m_2^2} - \frac{173\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{48m_1^2m_2^2} + \frac{13(\mathbf{p}_2^2)^3}{8m_1^2m_2^3}. \tag{A4b}
 \end{aligned}$$

$$\begin{aligned}
 H_{441}(\mathbf{x}_a, \mathbf{p}_a) &= \frac{5027(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4}{384m_1^4} - \frac{22993(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2\mathbf{p}_1^2}{960m_1^2} - \frac{6695(\mathbf{p}_1^2)^2}{1152m_1^2} - \frac{3191(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{640m_1^3m_2} \\
 &+ \frac{28561(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2}{1920m_1^2m_2} + \frac{8777(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{384m_1^2m_2} + \frac{752969\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{28800m_1^3m_2} \\
 &- \frac{16481(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{960m_1^2m_2^2} + \frac{94433(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2}{4800m_1^2m_2^2} - \frac{103957(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{2400m_1^3m_2^2} \\
 &+ \frac{791(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{400m_1^2m_2^2} + \frac{26627(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2\mathbf{p}_2^2}{1600m_1^2m_2^2} - \frac{118261\mathbf{p}_1^2\mathbf{p}_2^2}{4800m_1^2m_2^2} + \frac{105(\mathbf{p}_2^2)^2}{32m_1^2m_2^2}, \tag{A4c}
 \end{aligned}$$

$$\begin{aligned}
 H_{442}(\mathbf{x}_a, \mathbf{p}_a) &= \left(\frac{2749\pi^2}{8192} - \frac{211189}{19200}\right) \frac{(\mathbf{p}_1^2)^2}{m_1^4} + \left(\frac{63347}{1600} - \frac{1059\pi^2}{1024}\right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2\mathbf{p}_1^2}{m_1^4} + \left(\frac{375\pi^2}{8192} - \frac{23533}{1280}\right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4}{m_1^4} \\
 &+ \left(\frac{10631\pi^2}{8192} - \frac{1918349}{57600}\right) \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{m_1^2m_2^2} + \left(\frac{13723\pi^2}{16384} - \frac{2492417}{57600}\right) \frac{\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2m_2^2} \\
 &+ \left(\frac{1411429}{19200} - \frac{1059\pi^2}{512}\right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2}{m_1^2m_2^2} + \left(\frac{248991}{6400} - \frac{6153\pi^2}{2048}\right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1^2m_2^2} \\
 &- \left(\frac{30383}{960} + \frac{36405\pi^2}{16384}\right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^2m_2^2} + \left(\frac{1243717}{14400} - \frac{40483\pi^2}{16384}\right) \frac{\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1^2m_2} \\
 &+ \left(\frac{2369}{60} + \frac{35655\pi^2}{16384}\right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^3m_2} + \left(\frac{43101\pi^2}{16384} - \frac{391711}{6400}\right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2}{m_1^3m_2} \\
 &+ \left(\frac{56955\pi^2}{16384} - \frac{1646983}{19200}\right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1^2m_2}, \tag{A4d}
 \end{aligned}$$

$$H_{421}(\mathbf{x}_a, \mathbf{p}_a) = \frac{64861\mathbf{p}_1^2}{4800m_1^2} - \frac{91(\mathbf{p}_1 \cdot \mathbf{p}_2)}{8m_1m_2} + \frac{105\mathbf{p}_2^2}{32m_2^2} - \frac{9841(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2}{1600m_1^2} - \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{2m_1m_2}, \tag{A4e}$$

$$\begin{aligned}
 H_{422}(\mathbf{x}_a, \mathbf{p}_a) &= \left(\frac{1937033}{57600} - \frac{199177\pi^2}{49152}\right) \frac{\mathbf{p}_1^2}{m_1^2} + \left(\frac{176033\pi^2}{24576} - \frac{2864917}{57600}\right) \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1m_2} + \left(\frac{282361}{19200} - \frac{21837\pi^2}{8192}\right) \frac{\mathbf{p}_2^2}{m_2^2} \\
 &+ \left(\frac{698723}{19200} + \frac{21745\pi^2}{16384}\right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2}{m_1^2} + \left(\frac{63641\pi^2}{24576} - \frac{2712013}{19200}\right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1m_2} \\
 &+ \left(\frac{3200179}{57600} - \frac{28691\pi^2}{24576}\right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_2^2}, \tag{A4f}
 \end{aligned}$$

$$H_{40}(\mathbf{x}_a, \mathbf{p}_a) = -\frac{m_1^4}{16} + \left(\frac{6237\pi^2}{1024} - \frac{169799}{2400}\right) m_1^3m_2 + \left(\frac{44825\pi^2}{6144} - \frac{609427}{7200}\right) m_1^2m_2^2. \tag{A4g}$$

$$\begin{aligned}
 H_{4\text{PN}}^{\text{nonloc}}(t) &= -\frac{1}{5} \frac{G^2M}{c^8} I_{ij}^{(3)}(t) \\
 &\times \text{Pf}_{2r_{12}/c} \int_{-\infty}^{+\infty} \frac{dv}{|v|} I_{ij}^{(3)}(t+v),
 \end{aligned}$$

Analytical GW Templates for BBH Coalescences ?

PN corrections to Einstein's quadrupole frequency « chirping »
 from PN-improved balance equation $dE(f)/dt = -F(f)$

$$\frac{d\phi}{d \ln f} = \frac{\omega^2}{d\omega/dt} = Q_\omega^N \hat{Q}_\omega$$

$$Q_\omega^N = \frac{5c^5}{48\nu v^5}; \hat{Q}_\omega = 1 + c_2 \left(\frac{v}{c}\right)^2 + c_3 \left(\frac{v}{c}\right)^3 + \dots$$

$$\frac{v}{c} = \left(\frac{\pi G(m_1 + m_2) f}{c^3} \right)^{\frac{1}{3}}$$

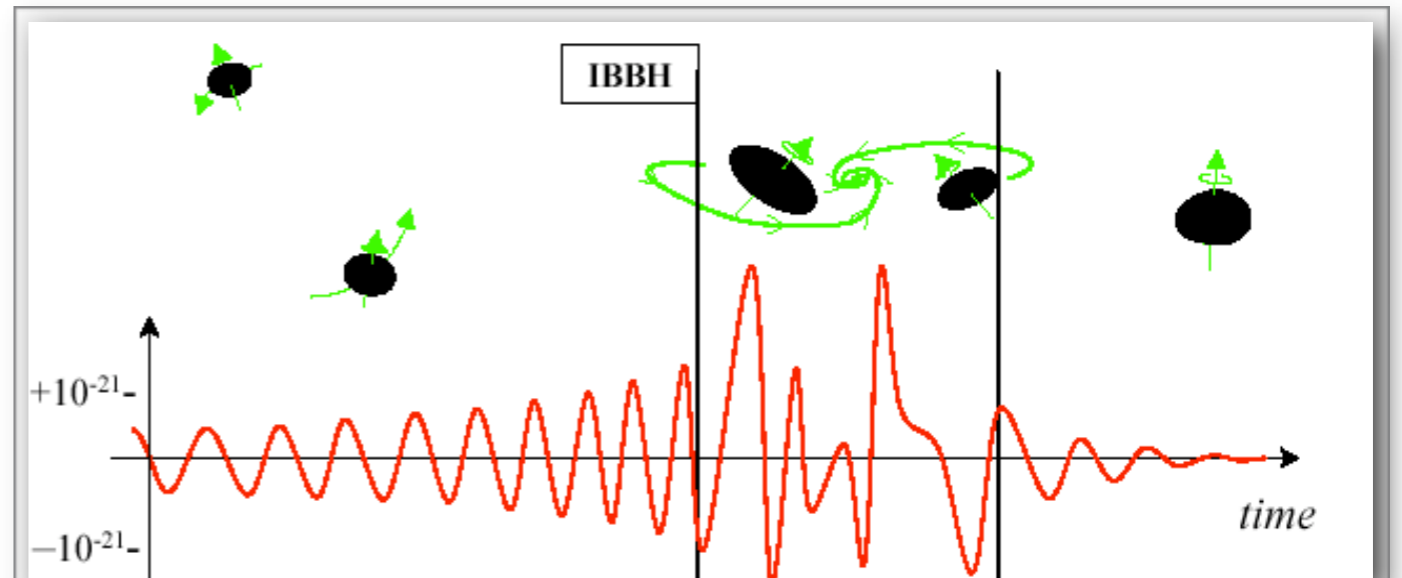
$$\nu = \frac{m_1 m_2}{(m_1 + m_2)^2}$$

Cutler et al. '93:

« slow convergence of PN »

Brady-Creighton-Thorne'98:

« inability of current computational techniques to evolve a BBH through its last ~10 orbits of inspiral » and to compute the merger



Damour-Iyer-Sathyaprakash'98:

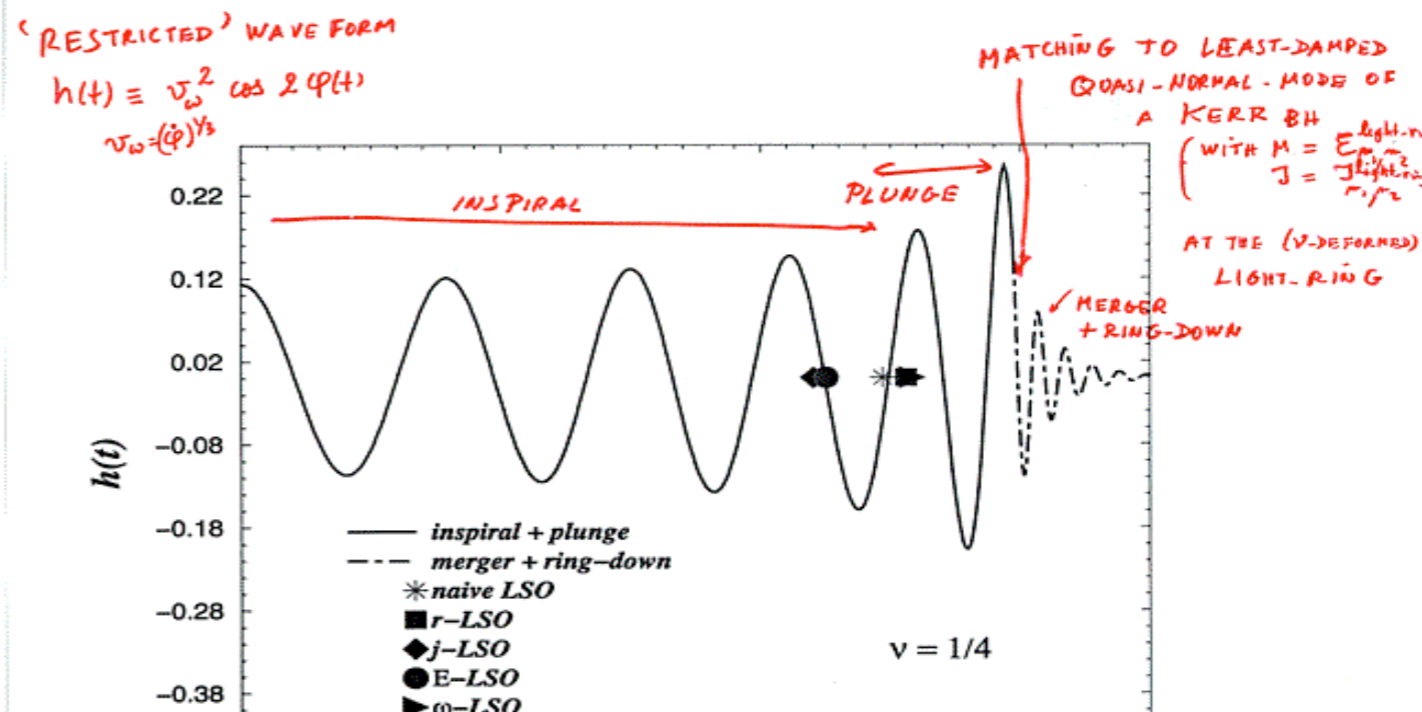
use **resummation** methods for E and F

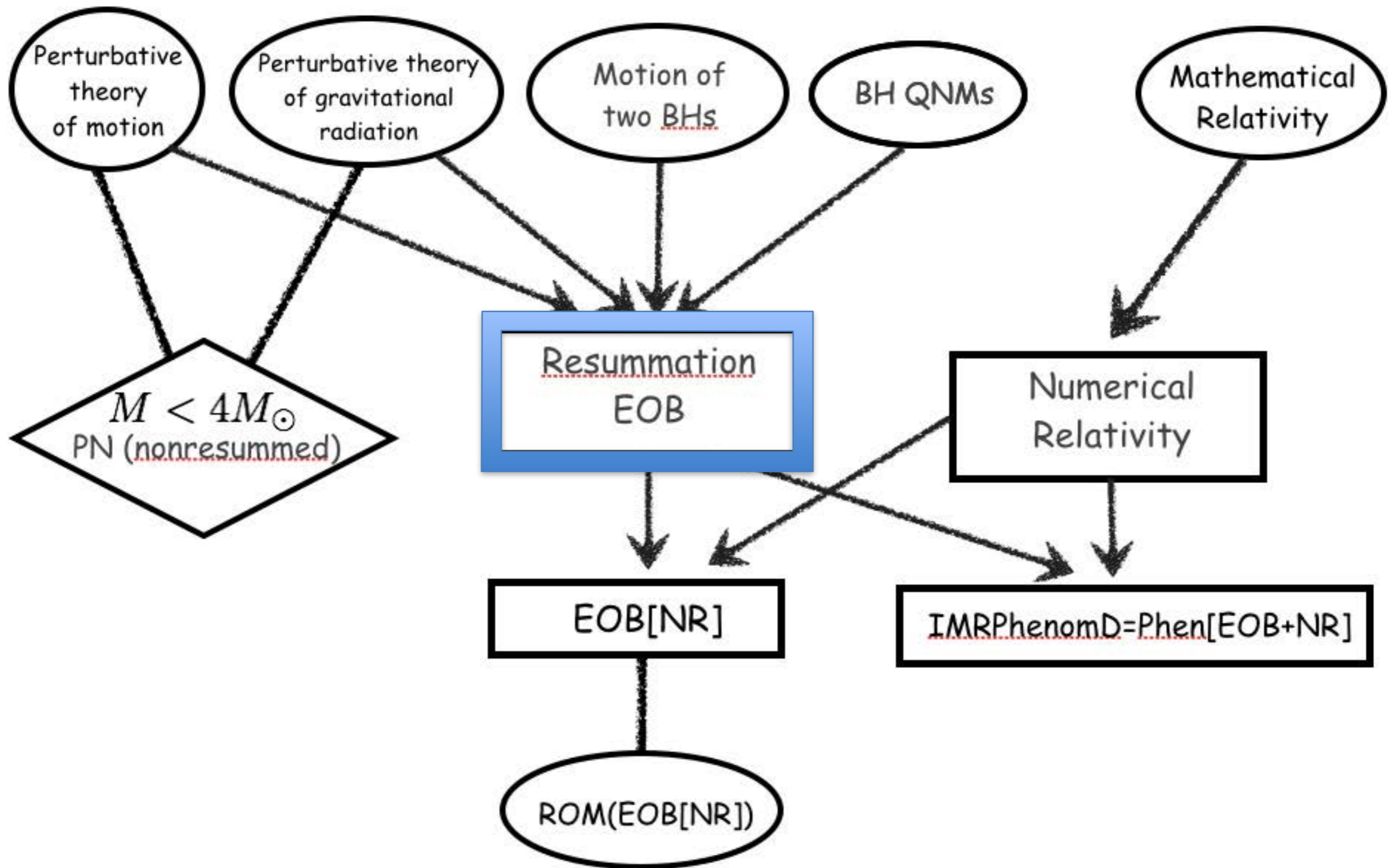
Buonanno-Damour '99-00:

novel, resummed approach:

Effective-One-Body

analytical formalism

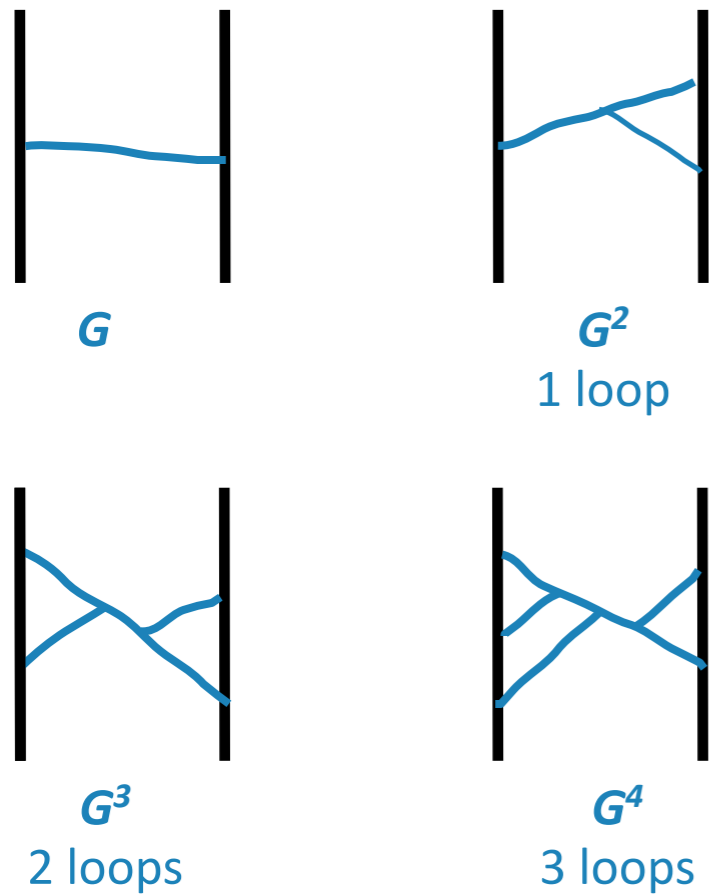




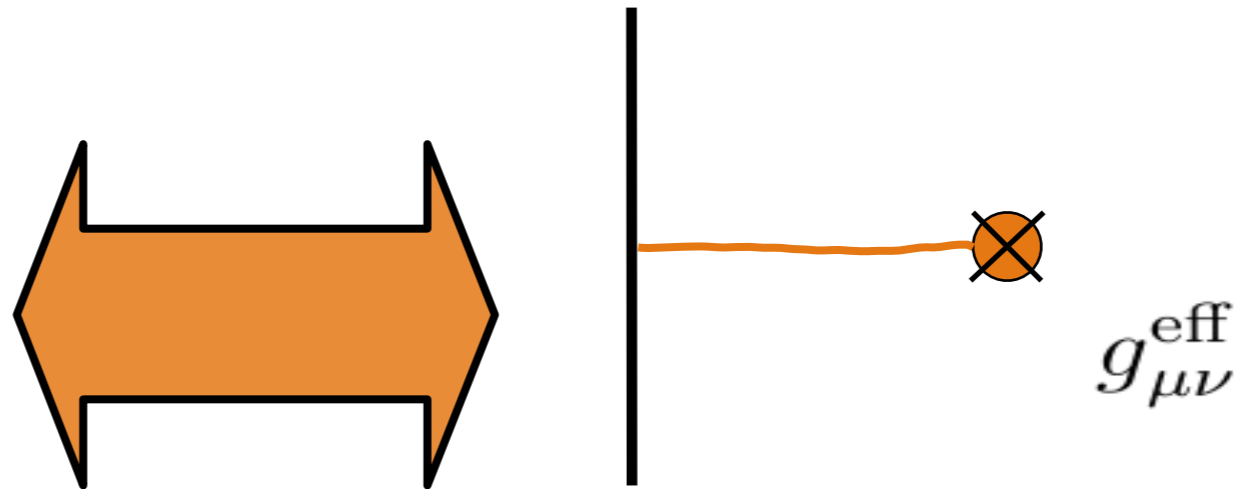
EOB formalism: Buonanno-Damour '99'00, Damour-Jaranowski-Schaefer '00, Damour '01, Damour-Iyer-Nagar '09

Real dynamics versus Effective dynamics

Real dynamics



Effective dynamics



$$S = - \int \mu ds + \dots$$

$$H = H_0 + \left(GH_1 + \frac{G^2}{c^2} H_2 + \frac{G^3}{c^4} H_3 + \frac{G^4}{c^6} H_4 \right) \left(1 + \frac{1}{c^2} + \dots \right)$$

Effective metric for non-spinning bodies: a nu-deformation of Schwarzschild

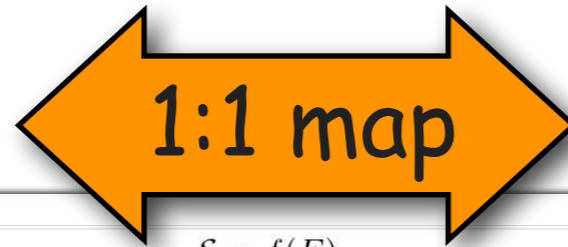
$$\nu = \frac{m_1 m_2}{(m_1 + m_2)^2}$$

$$ds_{\text{eff}}^2 = -A(r; \nu) dt^2 + B(r; \nu) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$$

TWO-BODY/EOB "CORRESPONDENCE":

THINK QUANTUM-MECHANICALLY (J.A. WHEELER)

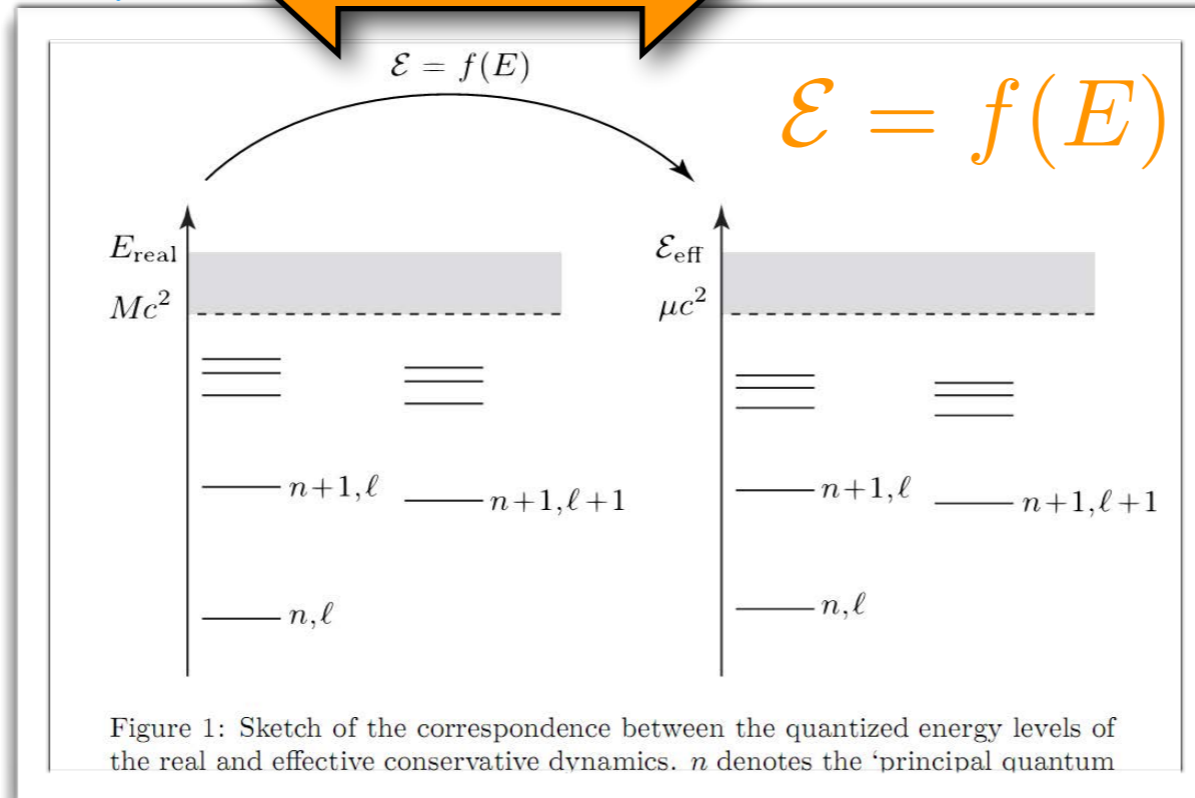
Real 2-body system
(in the c.o.m. frame)
 (m_1, m_2)



An effective particle
in some effective metric

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$g_{\mu\nu}^{\text{eff}}$$



$$\mu^2 + g_{\text{eff}}^{\mu\nu} \frac{\partial S_{\text{eff}}}{\partial x^\mu} \frac{\partial S_{\text{eff}}}{\partial x^\nu} + \mathcal{O}(p^4) = 0$$

Bohr-Sommerfeld's
Quantization Conditions
(action-angle variables &
Delaunay Hamiltonian)

$$J = l\hbar = \frac{1}{2\pi} \oint p_\varphi d\varphi$$

$$N = n\hbar = I_r + J$$

$$I_r = \frac{1}{2\pi} \oint p_r dr$$

$$H^{\text{classical}}(q, p) \longrightarrow H^{\text{classical}}(I_a) \longrightarrow E^{\text{quantum}}(I_a = n_a h) = f^{-1}[\mathcal{E}_{\text{eff}}^{\text{quantum}}(I_a^{\text{eff}} = n_a h)]$$

Resummed (non-spinning) EOB Hamiltonian

$$ds_{\text{eff}}^2 = -A(r; \nu) dt^2 + B(r; \nu) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$$

$$H_{\text{EOB}} = M \sqrt{1 + 2\nu \left(\frac{1}{\mu} \sqrt{A(r) \left(\mu^2 + \frac{p_r^2}{B(r)} + \frac{p_\phi^2}{r^2} + 2\nu(4 - 3\nu) \left(\frac{GM}{r} \right)^2 \frac{p_r^4}{\mu^2} \right) - 1} \right)}$$

$$A^{3PN}(r; M, \nu) = \text{Pade}_3^1 \left[1 - 2 \frac{GM}{c^2 r} + 2\nu \left(\frac{GM}{c^2 r} \right)^3 + \left(\frac{94}{3} - \frac{41}{32} \pi^2 \right) \nu \left(\frac{GM}{c^2 r} \right)^4 \right]$$

$$M = m_1 + m_2, \quad \mu = \frac{m_1 m_2}{m_1 + m_2}, \quad \nu = \frac{m_1 m_2}{(m_1 + m_2)^2} = \frac{\mu}{M}$$

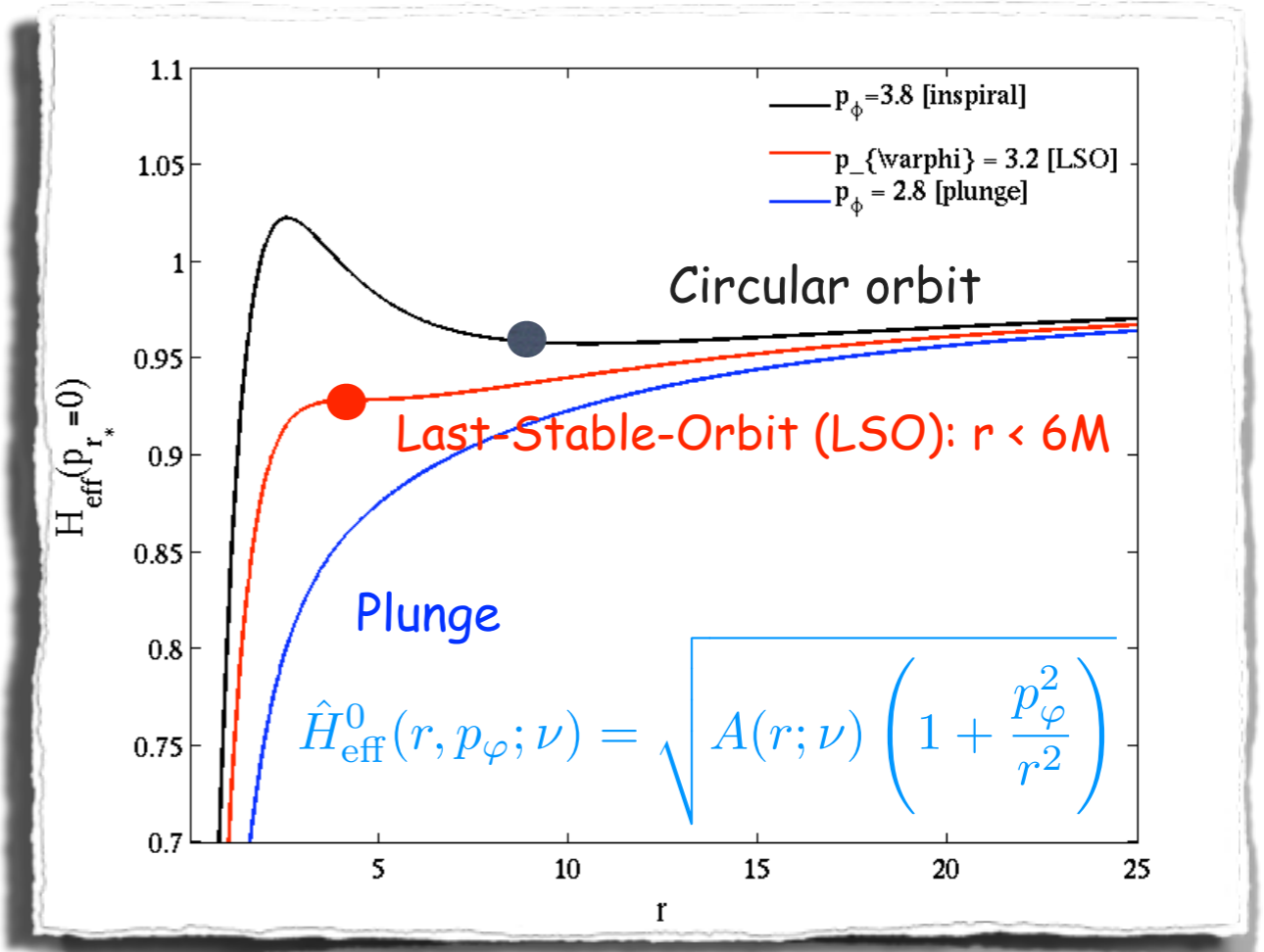
HAMILTON'S EQUATIONS & RADIATION REACTION

$$\dot{r} = \left(\frac{A}{B}\right)^{1/2} \frac{\partial \hat{H}_{\text{EOB}}}{\partial p_{r_*}}$$

$$\dot{\varphi} = \frac{\partial \hat{H}_{\text{EOB}}}{\partial p_{\varphi}} \equiv \Omega$$

$$\dot{p}_{r_*} = - \left(\frac{A}{B}\right)^{1/2} \frac{\partial \hat{H}_{\text{EOB}}}{\partial r} + \hat{\mathcal{F}}_{r_*}$$

$$\dot{p}_{\varphi} = \hat{\mathcal{F}}_{\varphi}$$

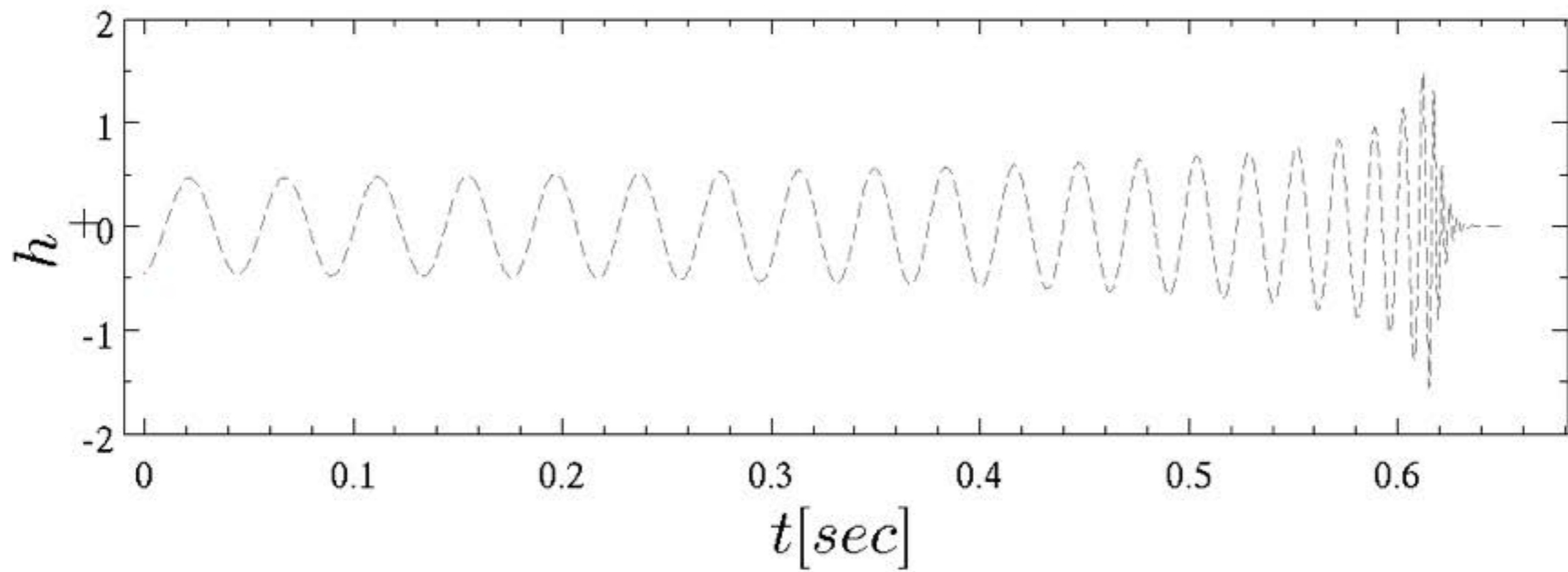
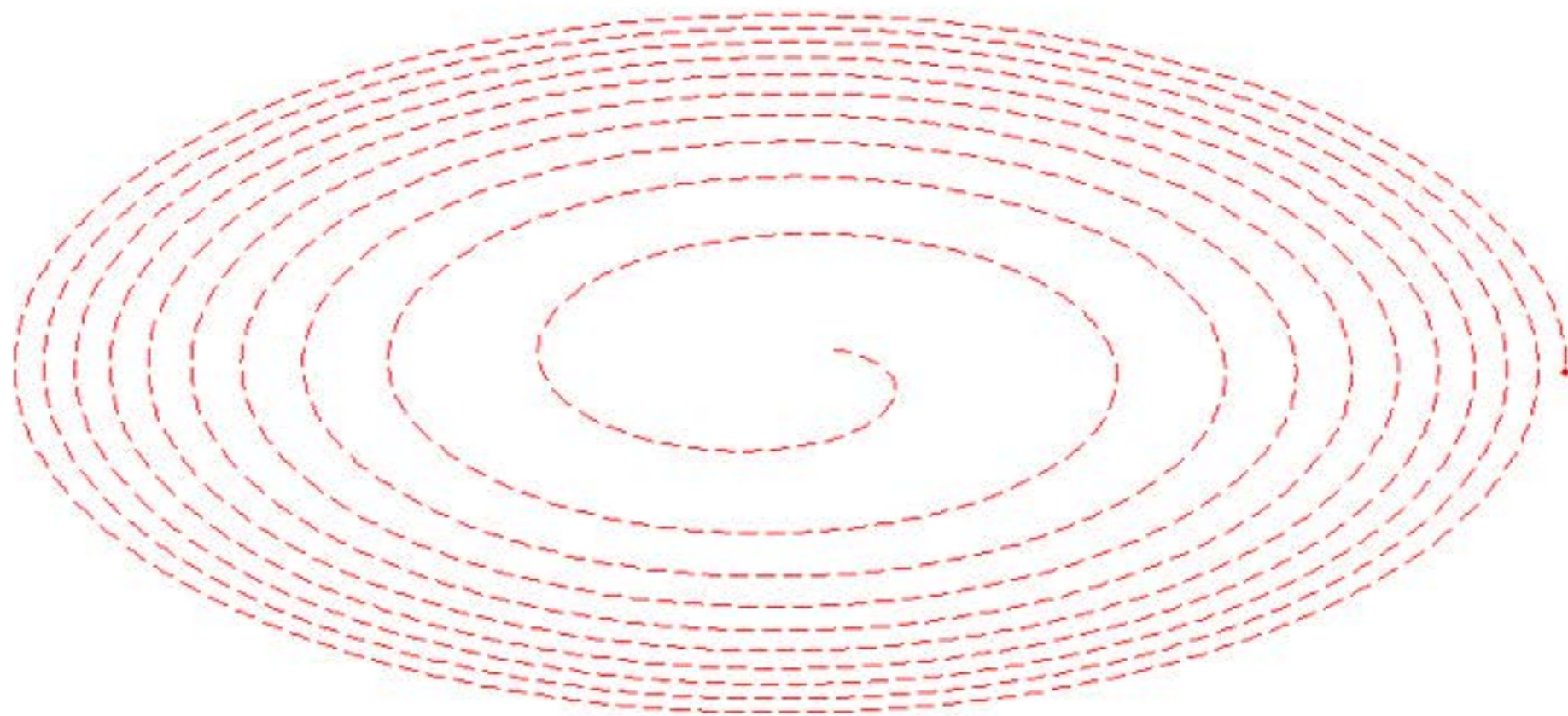


- ▶ The system must radiate angular momentum
- ▶ How? Use PN-based (Taylor-expanded) radiation reaction force (ang-mom flux)
- ▶ Need flux resummation

$$\hat{\mathcal{F}}_{\varphi}^{\text{Taylor}} = -\frac{32}{5} \nu \Omega^5 r_{\Omega}^4 \hat{F}^{\text{Taylor}}(v_{\varphi}) \longrightarrow$$

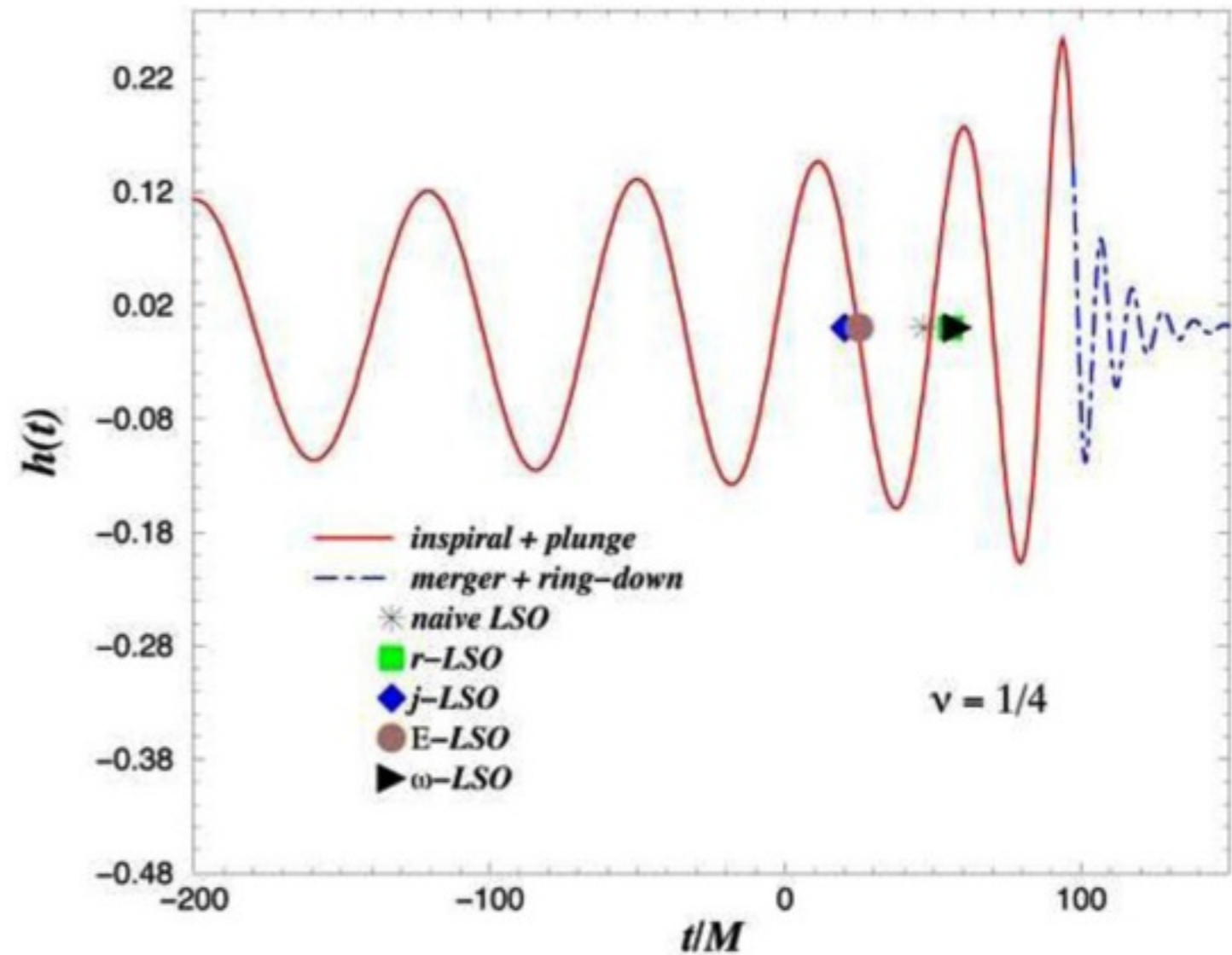
Plus horizon contribution [Nagar&Akcaay2012]

Resummation multipole by multipole
(Damour&Nagar 2007,
Damour, Iyer & Nagar 2008,
Damour & Nagar, 2009)



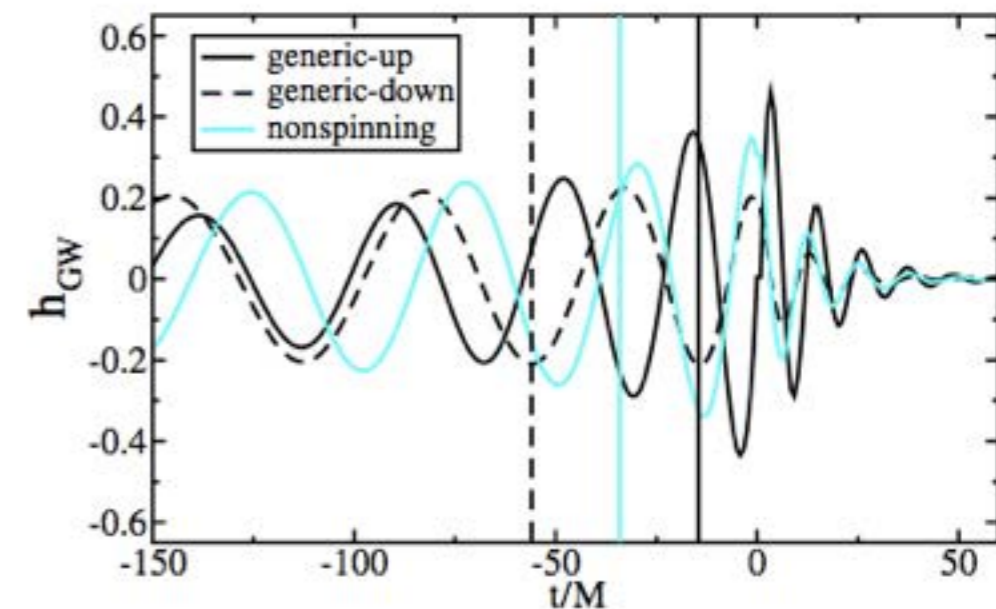
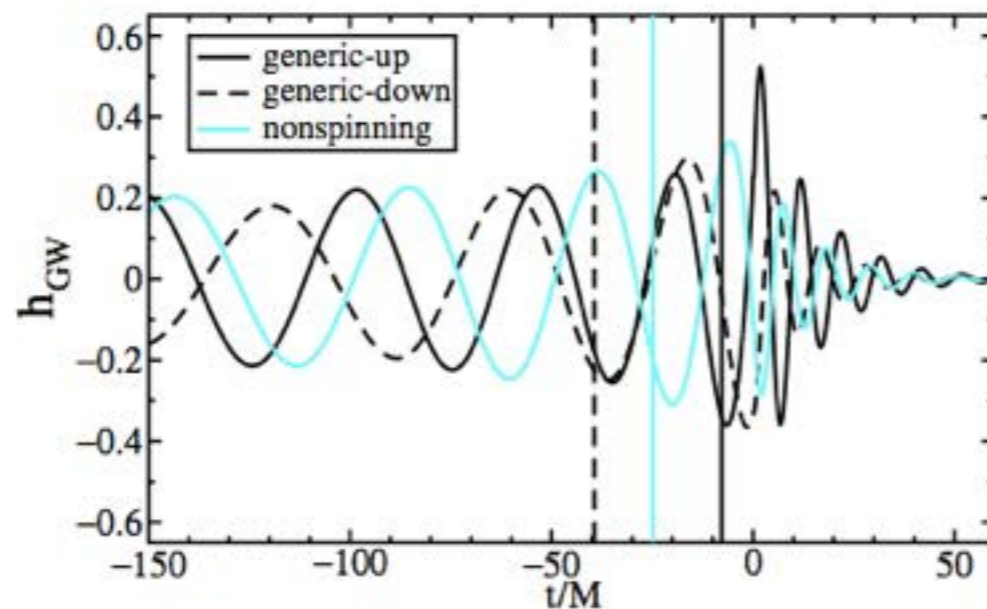
First complete waveforms for BBH coalescences: analytical EOB

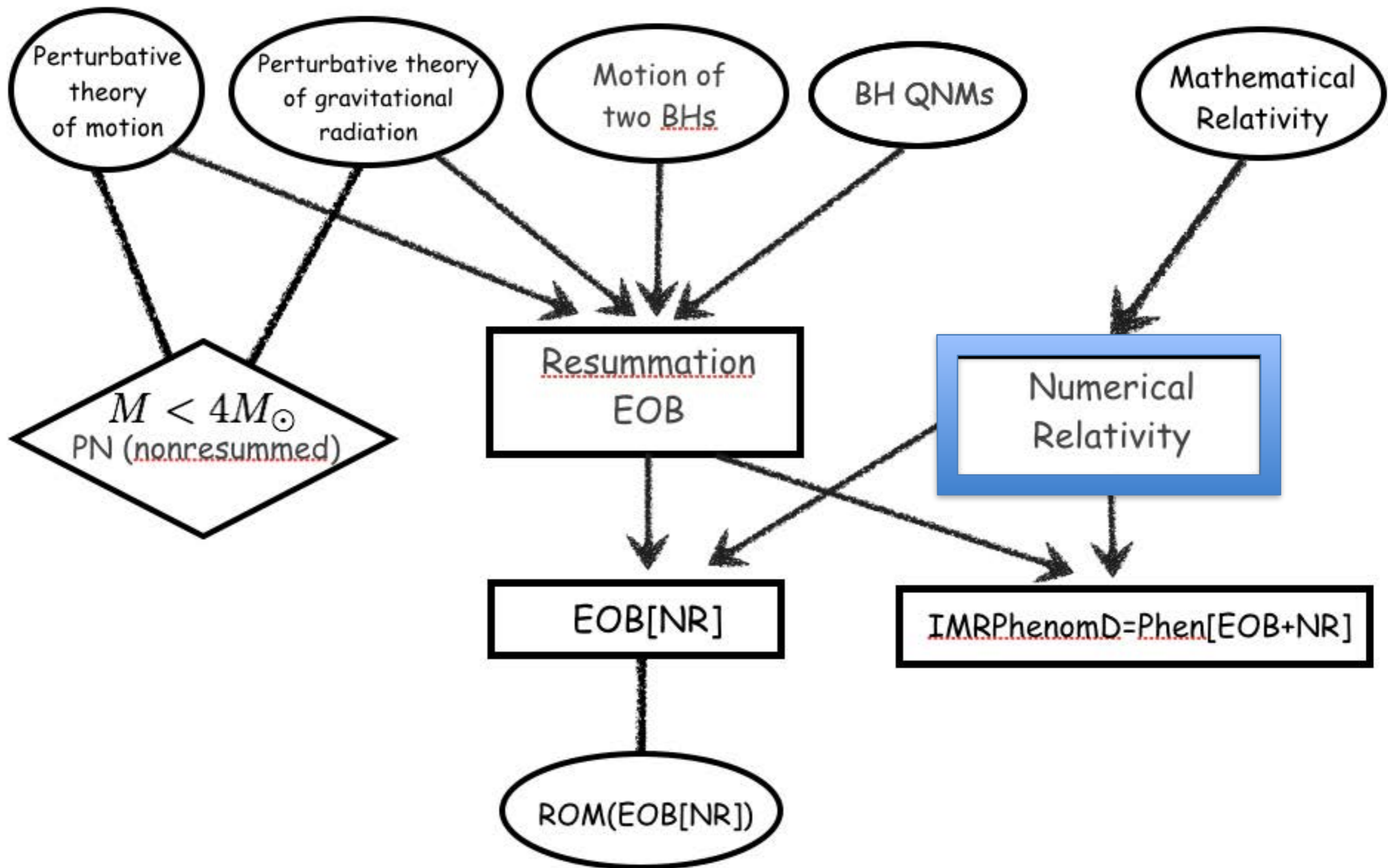
Non-spinning BHs
Buonanno-Damour 2000



Spinning BHs
Buonanno-Chen-Damour
Nov 2005:

« to show the
promise
of a purely
analytical
EOB-based
approach »





Numerical Relativity (NR)

Mathematical foundations : Darmois 27, Lichnerowicz 43, Choquet-Bruhat 52-

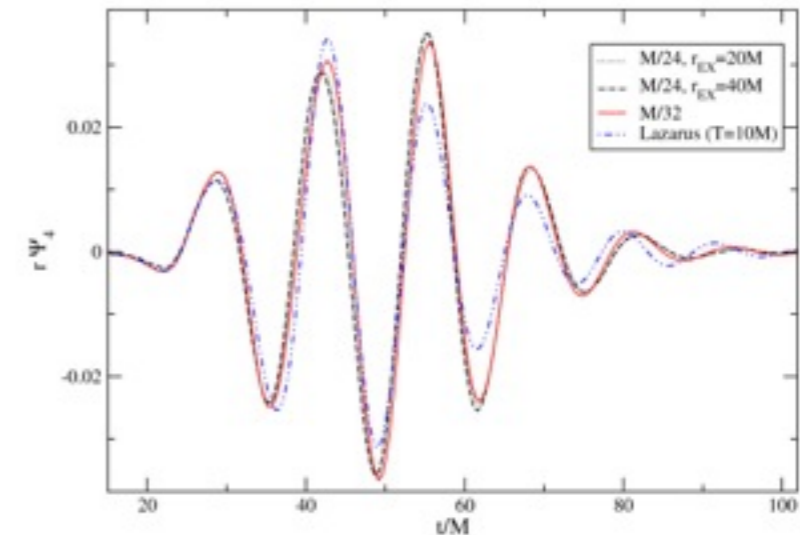
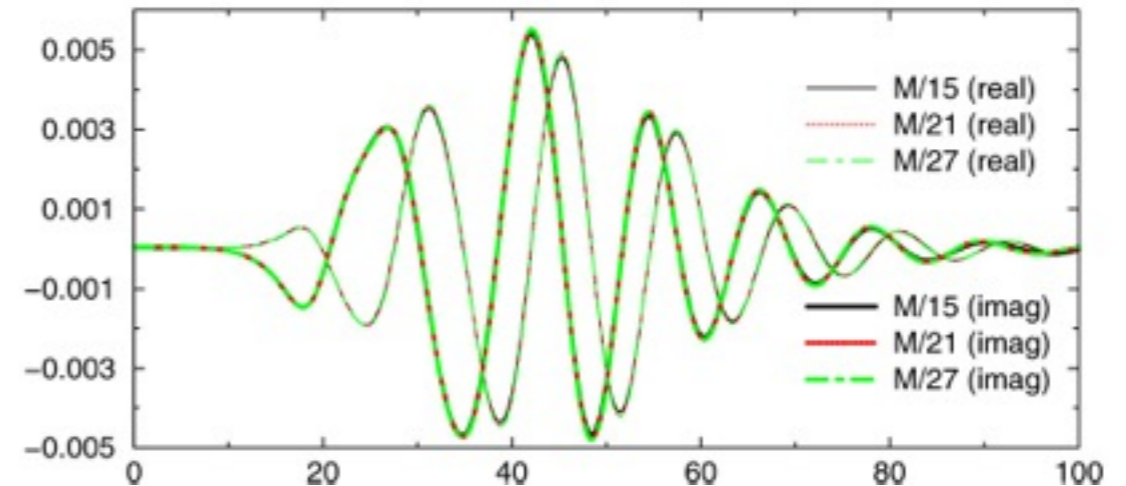
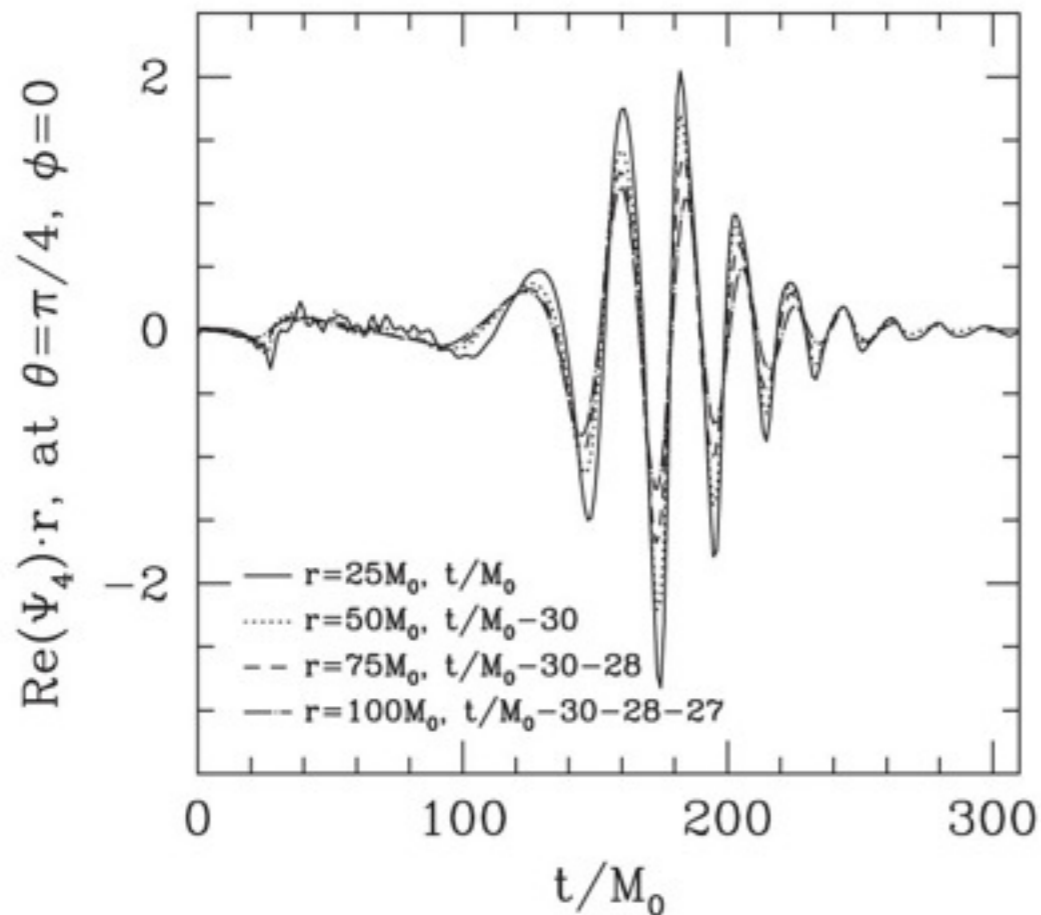
Breakthrough:

Pretorius 2005 generalized harmonic coordinates, constraint damping, excision

Campanelli-Lousto-Maronetti-Zlochover 2006

Baker-Centrella-Choi-Koppitz-van Meter 2006

Moving punctures



The first EOB vs NR comparison

Buonanno-Cook-Pretorius 2007

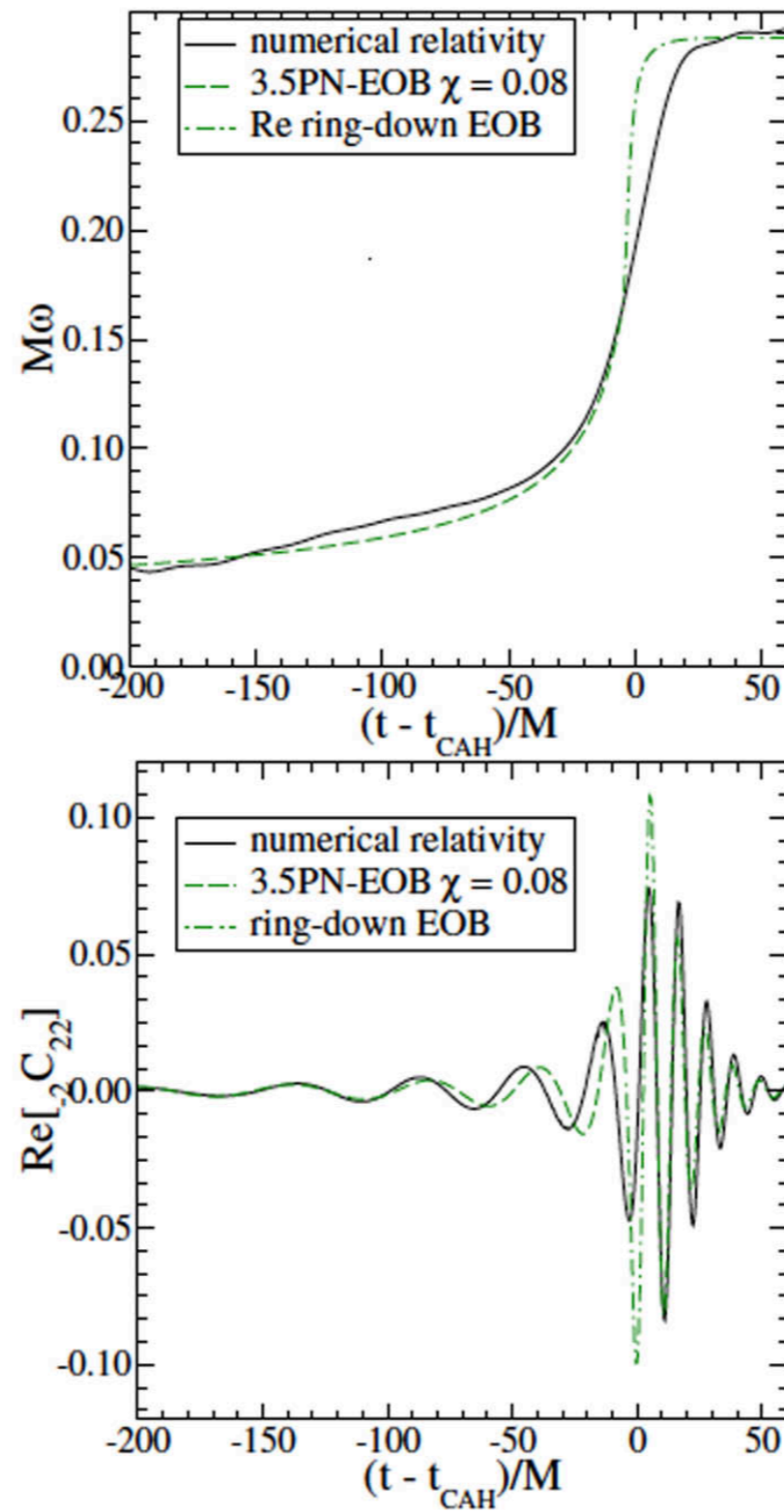
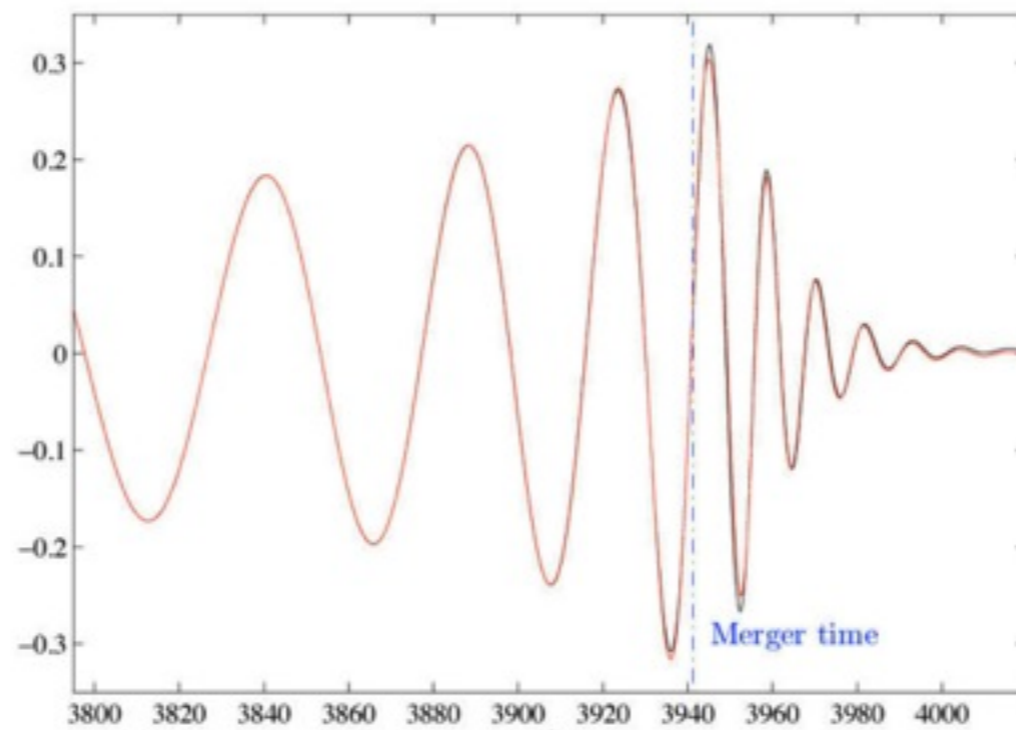
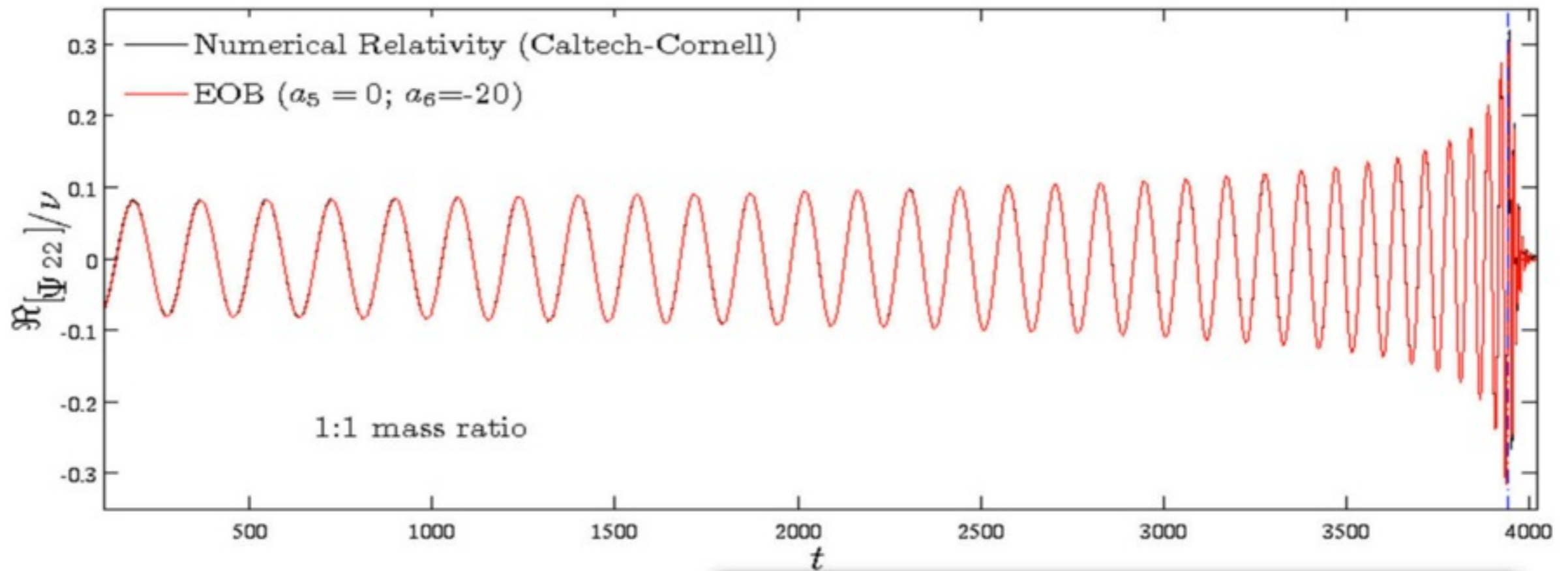
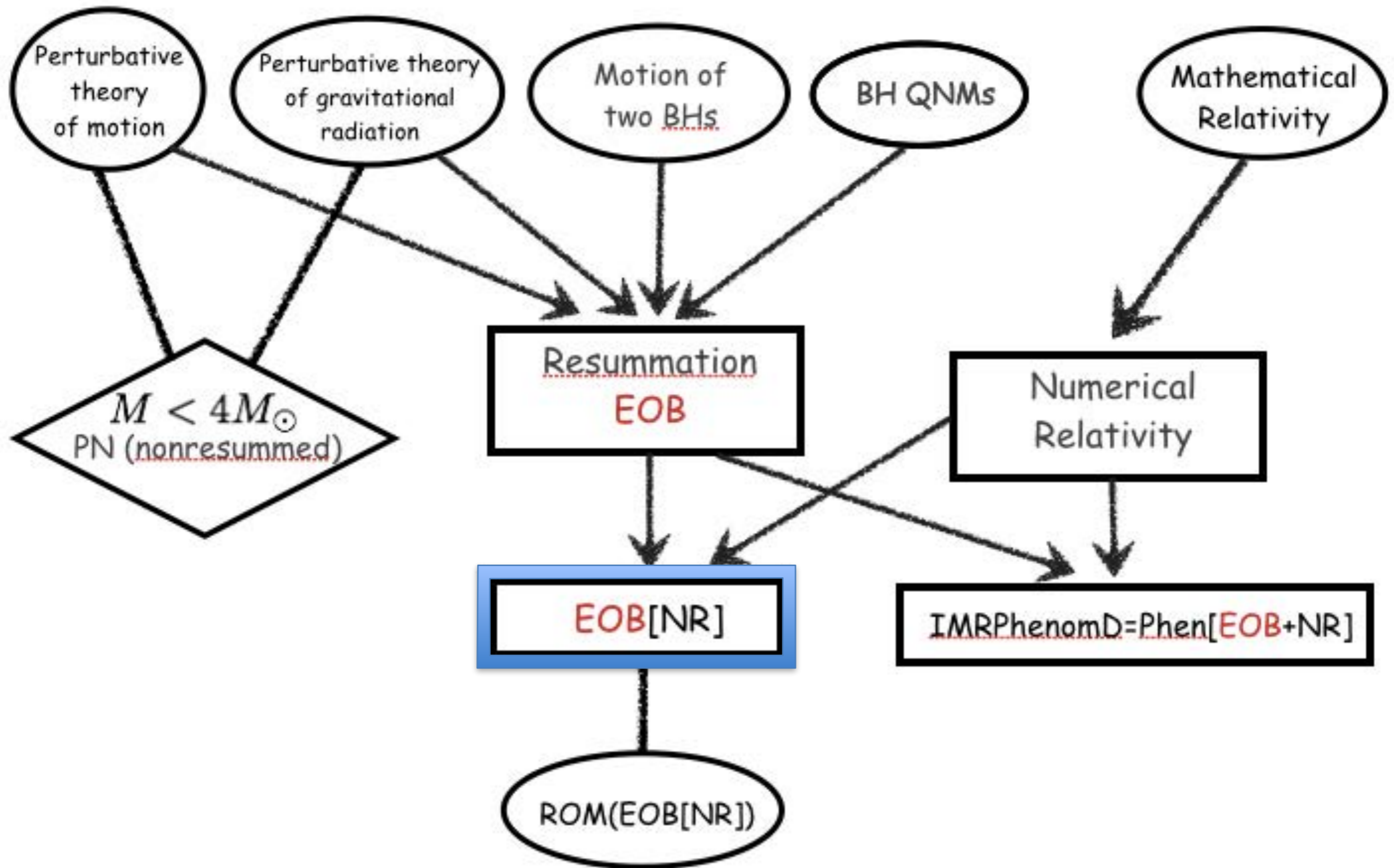


FIG. 21 (color online). We compare the NR and EOB frequency and $\text{Re}[-2C_{22}]$ waveforms throughout the entire inspiral–merger–ring-down evolution. The data refers to the $d = 16$ run.

Numerical Relativity Waveform (Caltech-Cornell, SXS)





EOB[NR]: Damour-Gourgoulhon-Grandclement '02, Damour-Nagar '07-16, Buonanno-Pan-Taracchini-....'07-16

NR-completed resummed 5PN EOB radial A potential

4PN analytically complete + 5 PN logarithmic term in the $A(u, \nu)$ function,

With $u = GM/R$ and $\nu = m_1 m_2 / (m_1 + m_2)^2$

[Damour 09, Blanchet et al 10, Barack-Damour-Sago 10, Le Tiec et al 11,

Barausse et al 11, Akcay et al 12, Bini-Damour 13,

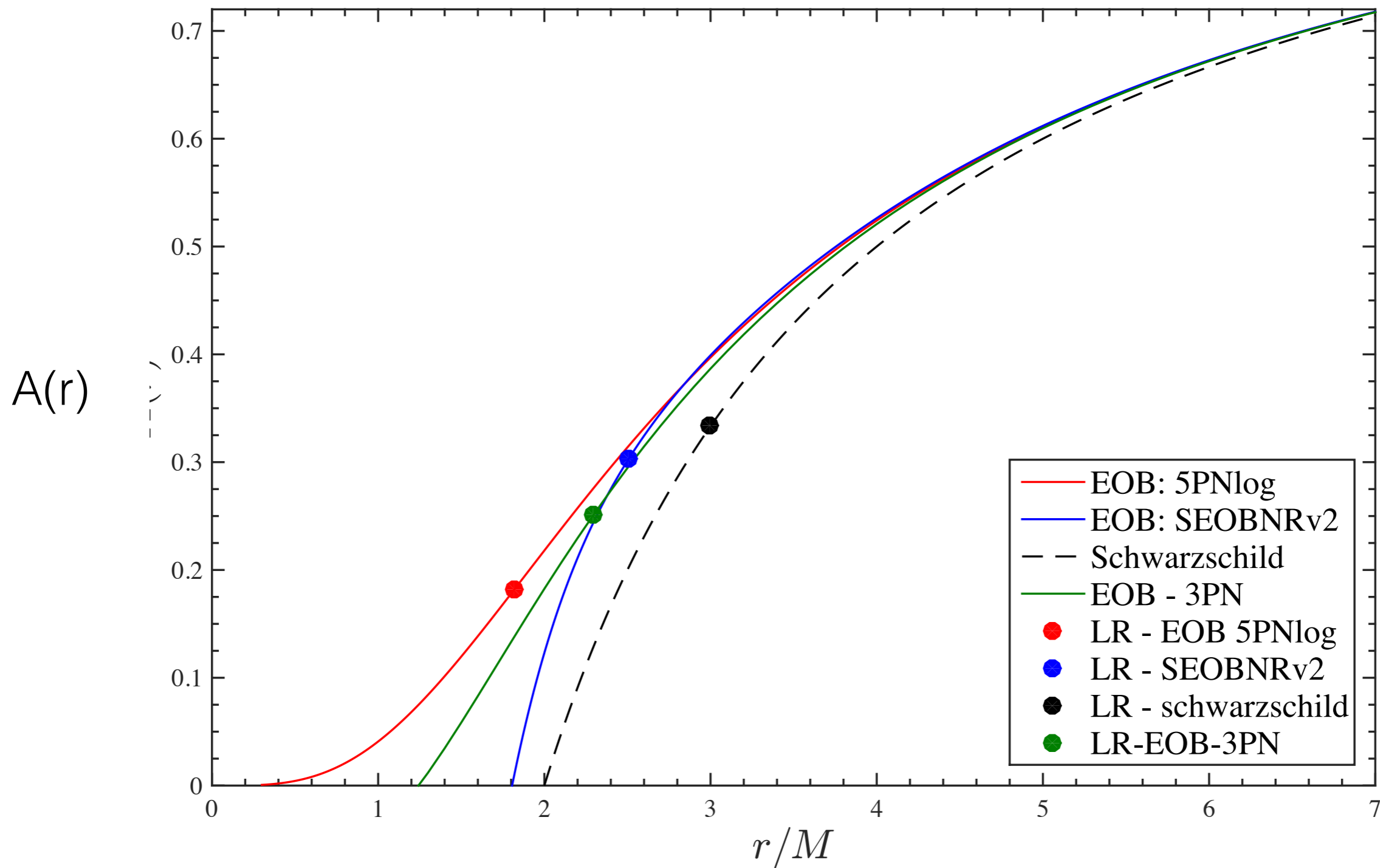
Damour-Jaranowski-Schäfer 14, Nagar-Damour-Reisswig-Pollney 15]

$$\begin{aligned} A(u; \nu, a_6^c) &= P_5^1 \left[1 - 2u + 2\nu u^3 + \nu \left(\frac{94}{3} - \frac{41}{32} \pi^2 \right) u^4 \right. \\ &+ \nu \left[-\frac{4237}{60} + \frac{2275}{512} \pi^2 + \left(-\frac{221}{6} + \frac{41}{32} \pi^2 \right) \nu + \frac{64}{5} \ln(16e^{2\gamma} u) \right] u^5 \\ &+ \left. \nu \left[a_6^c(\nu) - \left(\frac{7004}{105} + \frac{144}{5} \nu \right) \ln u \right] u^6 \right] \end{aligned}$$

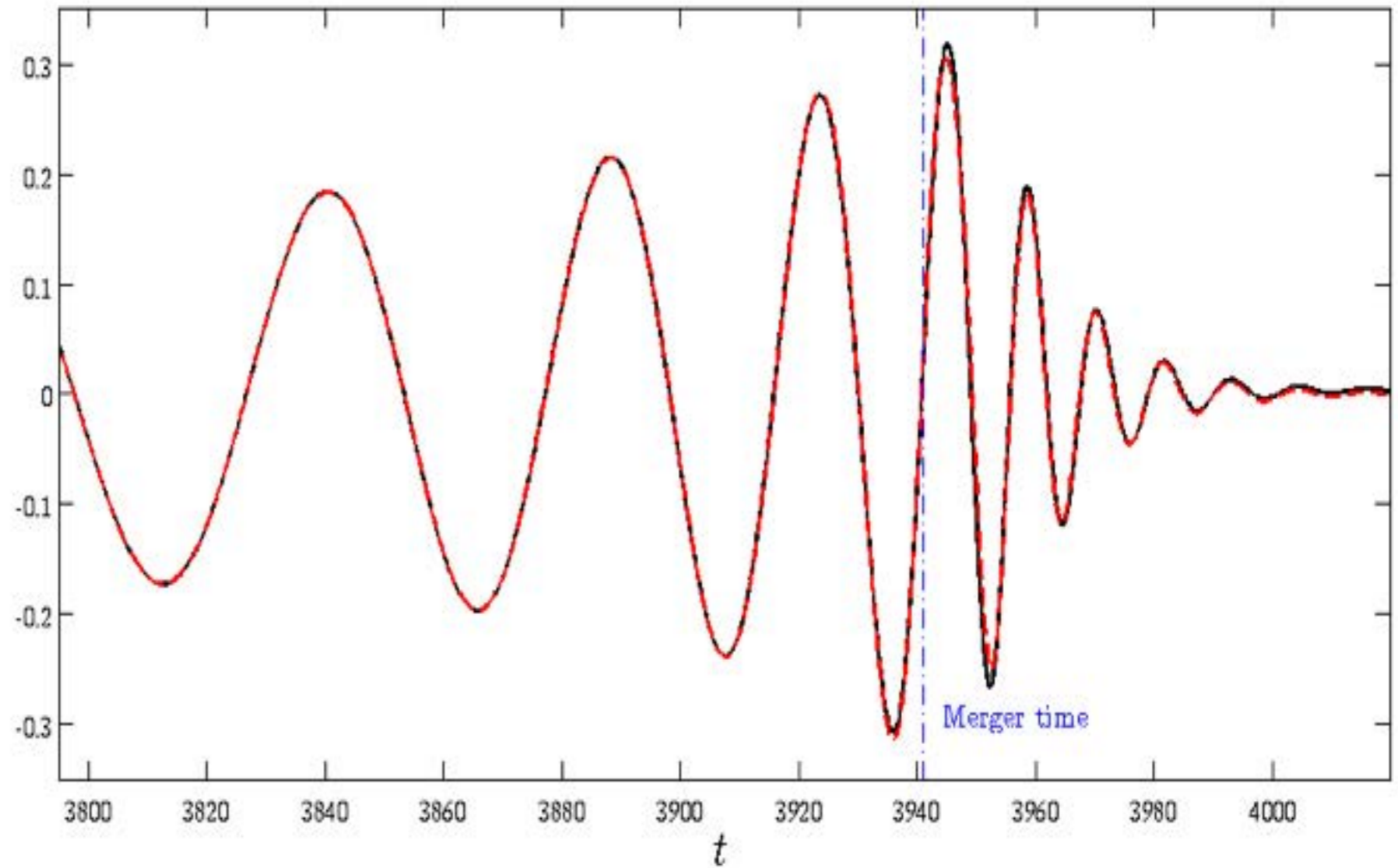
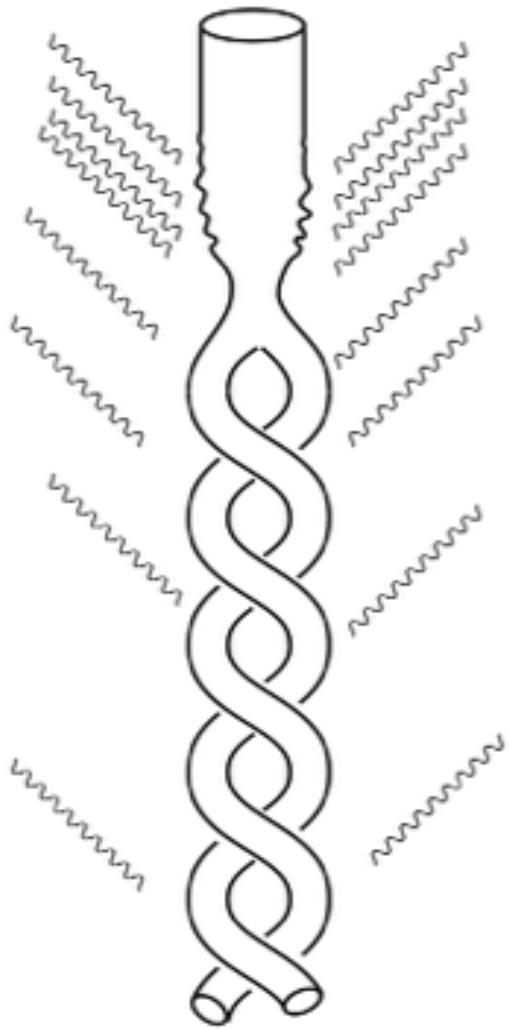
$$a_6^{c \text{ NR-tuned}}(\nu) = 81.38 - 1330.6 \nu + 3097.3 \nu^2$$

MAIN RADIAL RADIAL EOB POTENTIAL A(R)

m1=m2 case



EOB[NR] vs NR Comparison

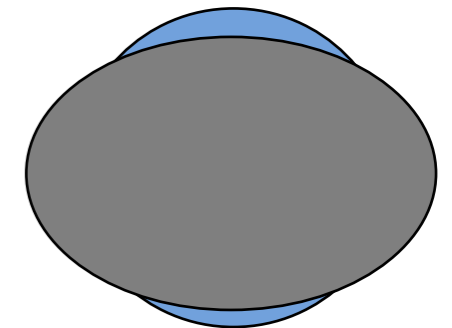


Inspiral + « plunge »



Two orbiting point-masses:
Resummed dynamics

Ringdown BH



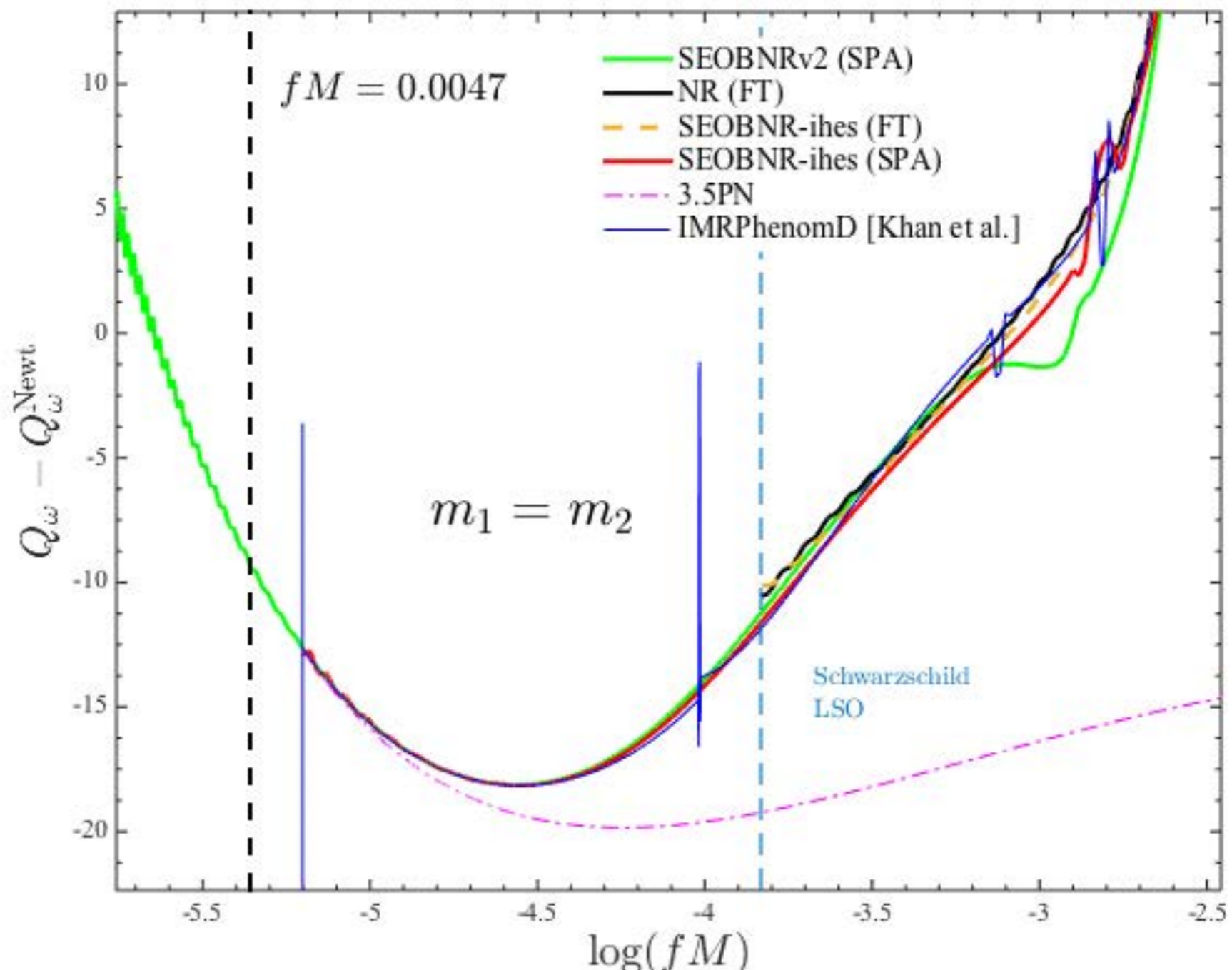
Peak emitted power $\sim 3 \times 10^{56}$ erg/s $\sim 0.001 c^5/G$

PN, EOB, NR, PHENOMD

PN accuracy loss during inspiral

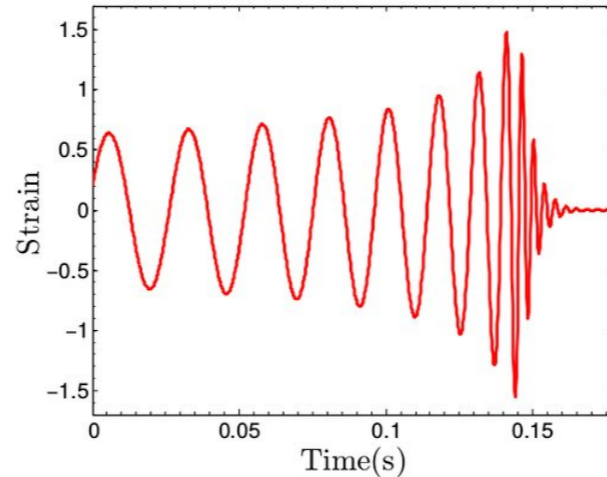
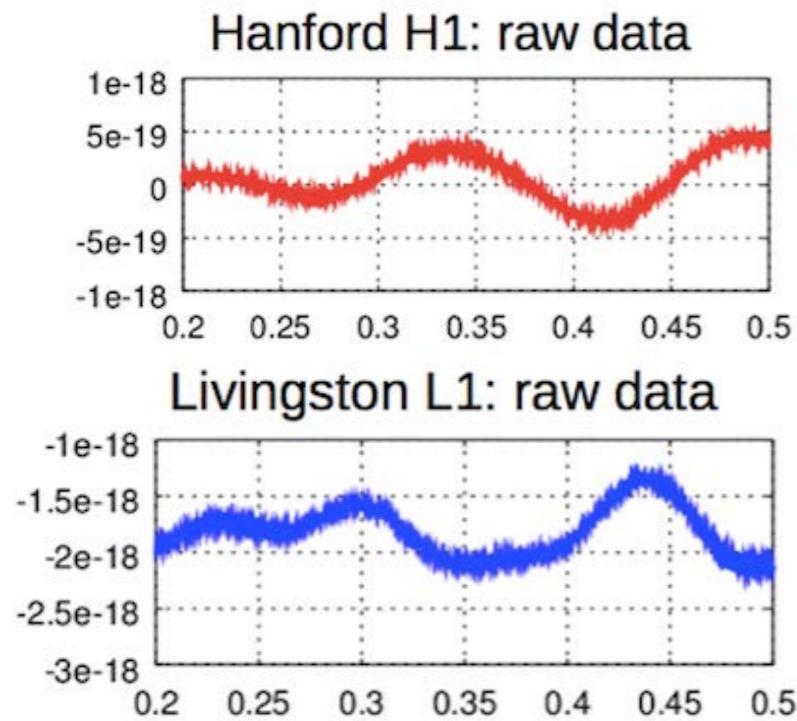
Dimensionless « quality factor » of GW phase $Q_\omega = f^2 \frac{d^2 \psi(f)}{df^2} \approx \frac{\omega^2}{\dot{\omega}}$

$$Q_\omega - Q_\omega^N$$



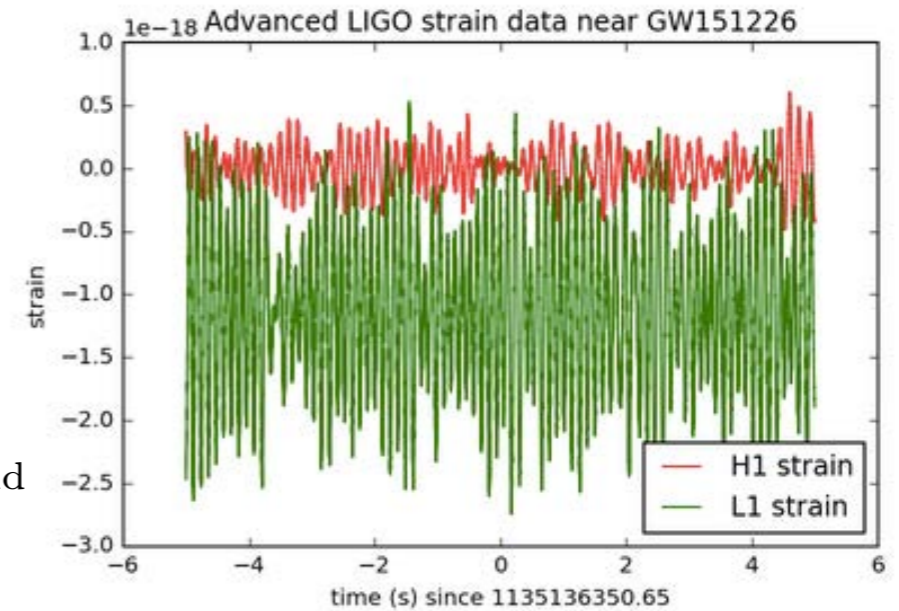
GW150914 and GW151226: incredibly small signals lost in the broad-band noise

GW150914, from Chassande-Mottin 5 April 2016



$$h_{GW}^{\max} \sim 10^{-21} \sim 10^{-3} h_{LIGO}^{\text{broadband}}$$

GW151226 from LIGO open data



Several levels of signal search and analysis:

1. **online time-frequency** analysis (Wilson, Meyer, Daubechies-Jaffard-Journe, Klimenko et al.)
2. **online matched-filter** analysis (EOB[NR] and Phenom[EOB+NR])
3. **offline matched-filter** analysis (significance, parameter estimation)(EOB[NR] templates with PyCBC or GstLAL)

Matched Filtering

$$\langle output | h_{\text{template}} \rangle = \int \frac{df}{S_n(f)} o(f) h_{\text{template}}^*(f)$$

MATCHED FILTERING SEARCH AND DATA ANALYSIS

Precomputed bank of 250 000 templates for inspiralling and coalescing BBH GW waveforms: m_1 , m_2 , $\chi_1=S_1/m_1^2$, $\chi_2=S_2/m_2^2$

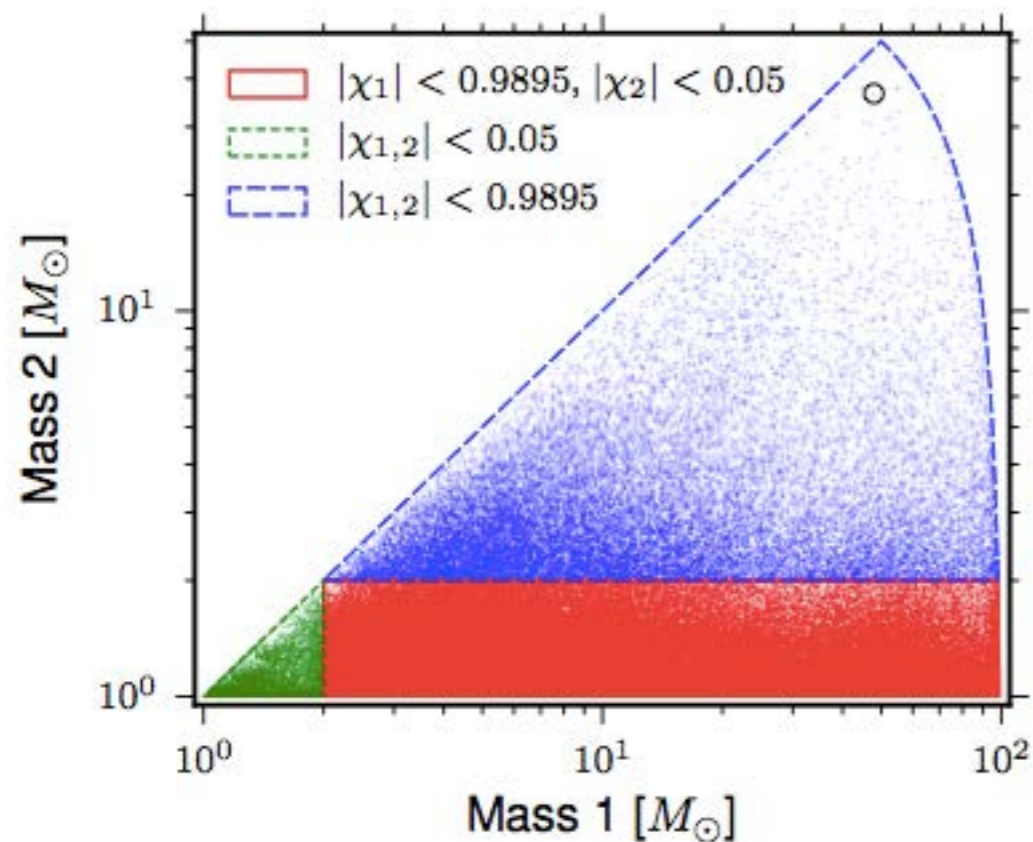
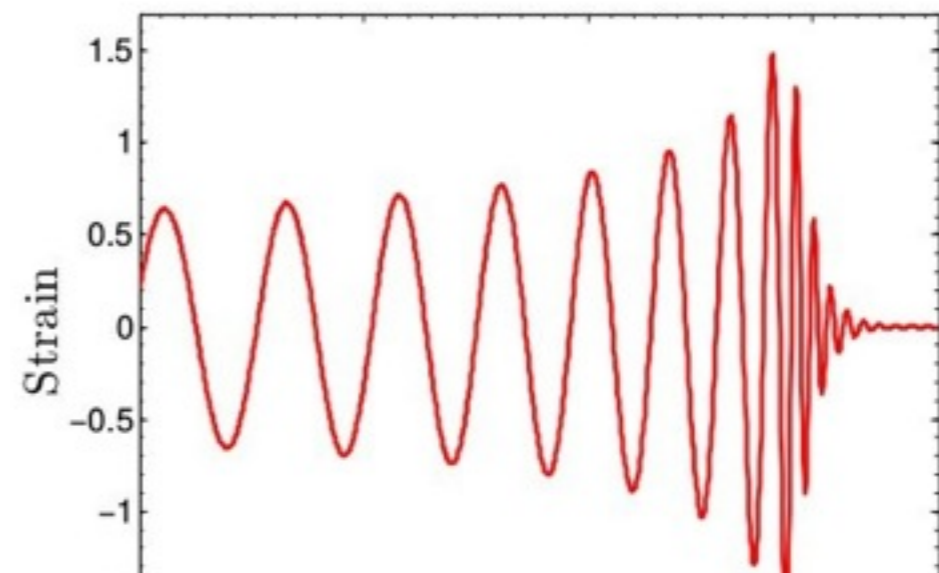


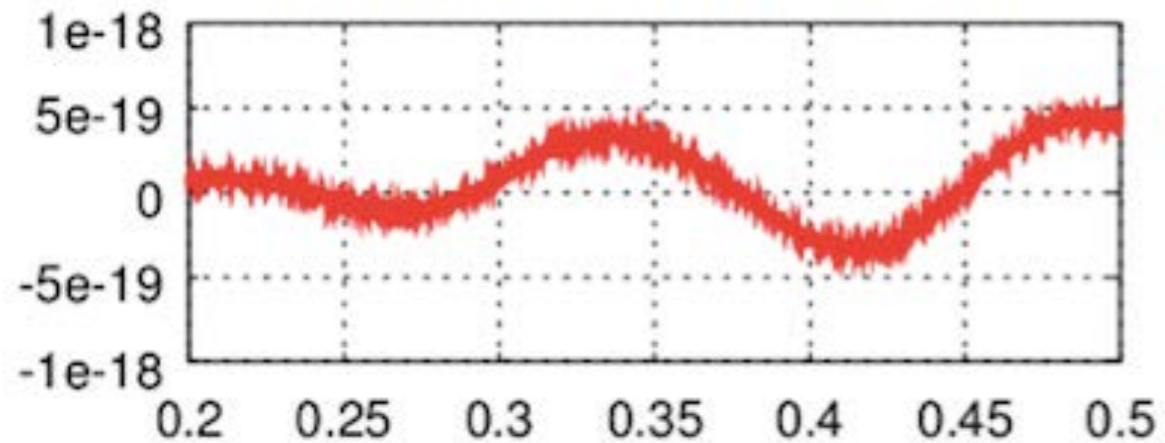
FIG. 1. The four-dimensional search parameter space covered by the template bank shown projected into the component-mass plane, using the convention $m_1 > m_2$. The lines bound mass regions with different limits on the dimensionless aligned-spin parameters χ_1 and χ_2 . Each point indicates the position of a template in the bank. The circle highlights the template that best matches GW150914. This does not coincide with the best-fit parameters due to the discrete nature of the template bank.

Search template bank made of **SpinningEOB[NR] templates** (Buonanno-Damour 99, Damour '01..., Taracchini et al. 14) in ROM form; **Recently improved** (Bohé et al '16') by including leading 4PN terms (Bini-Damour '13), spin-dependent terms (Pan-Buonanno et al. '13), and calibrating against 141 NR simulations.

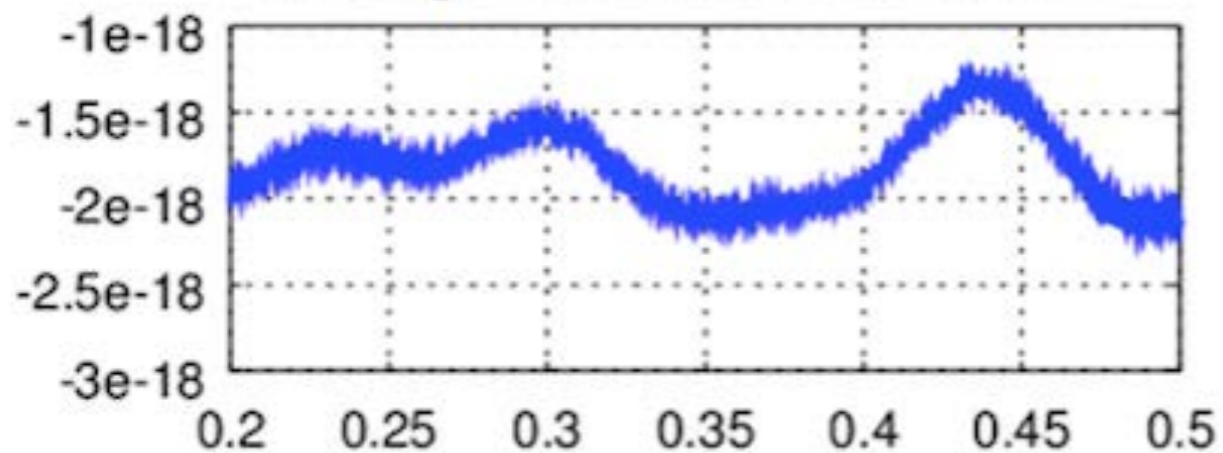


GW150914 vs EOB[NR]

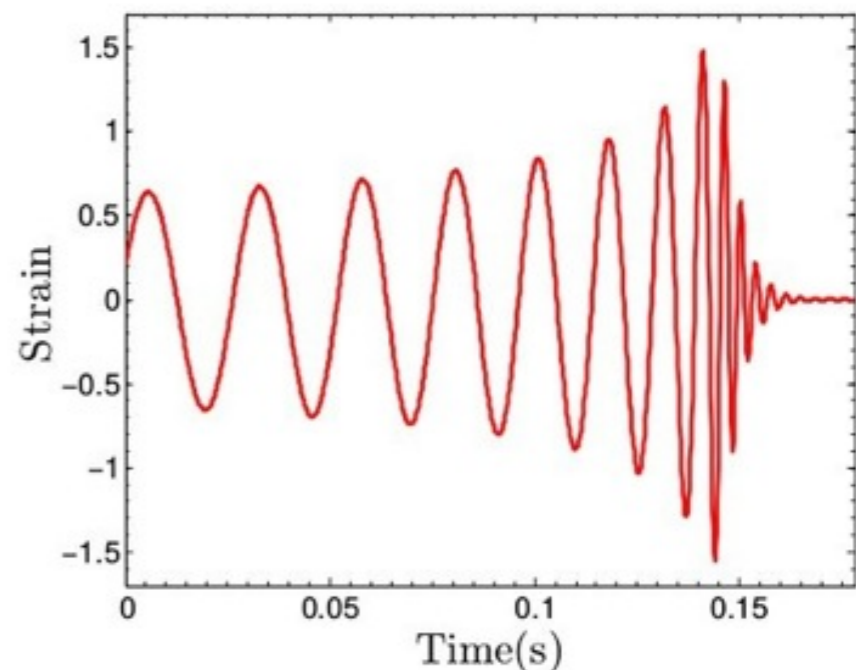
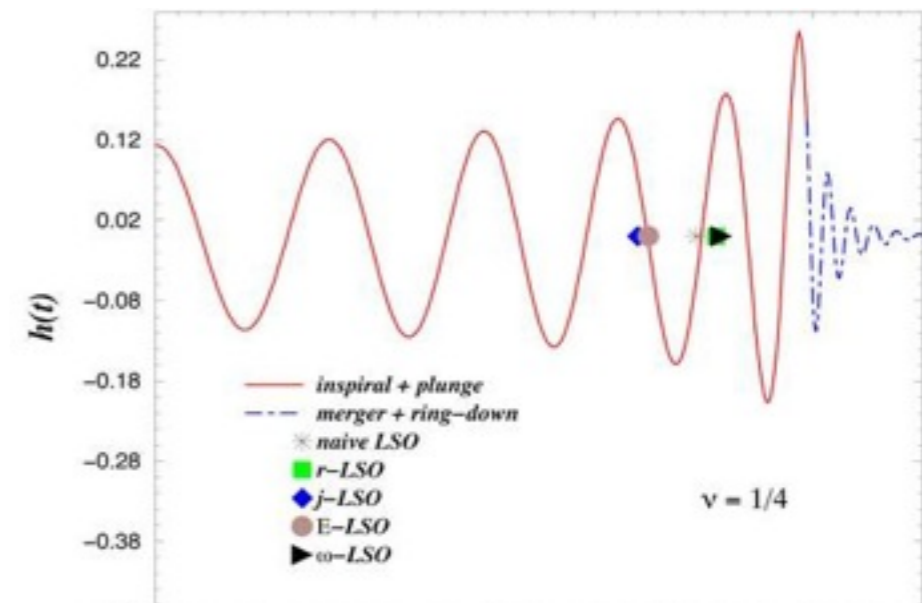
Hanford H1: raw data



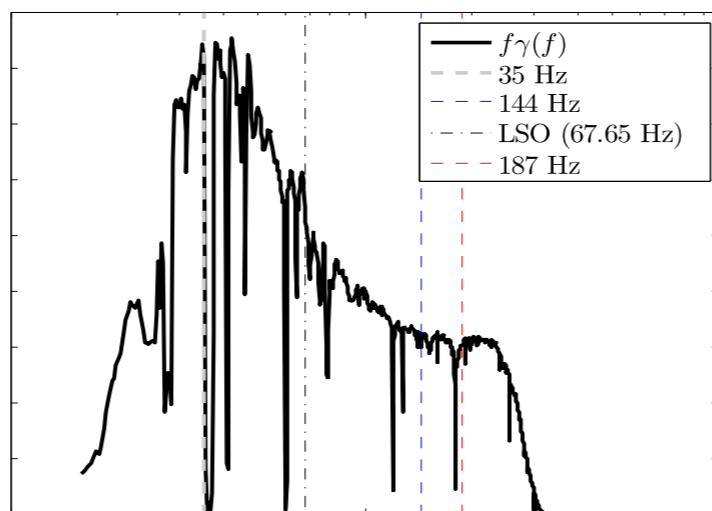
Livingston L1: raw data



scale : 10^{-21}
500 × smaller



$$\frac{d\rho^2}{d \ln f} = \frac{f |\tilde{h}(f)|^2}{S_n(f)}$$



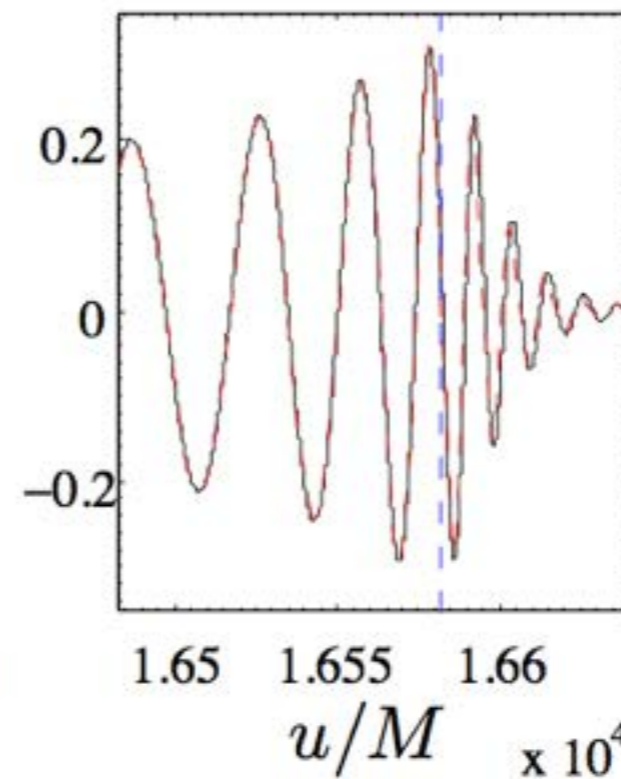
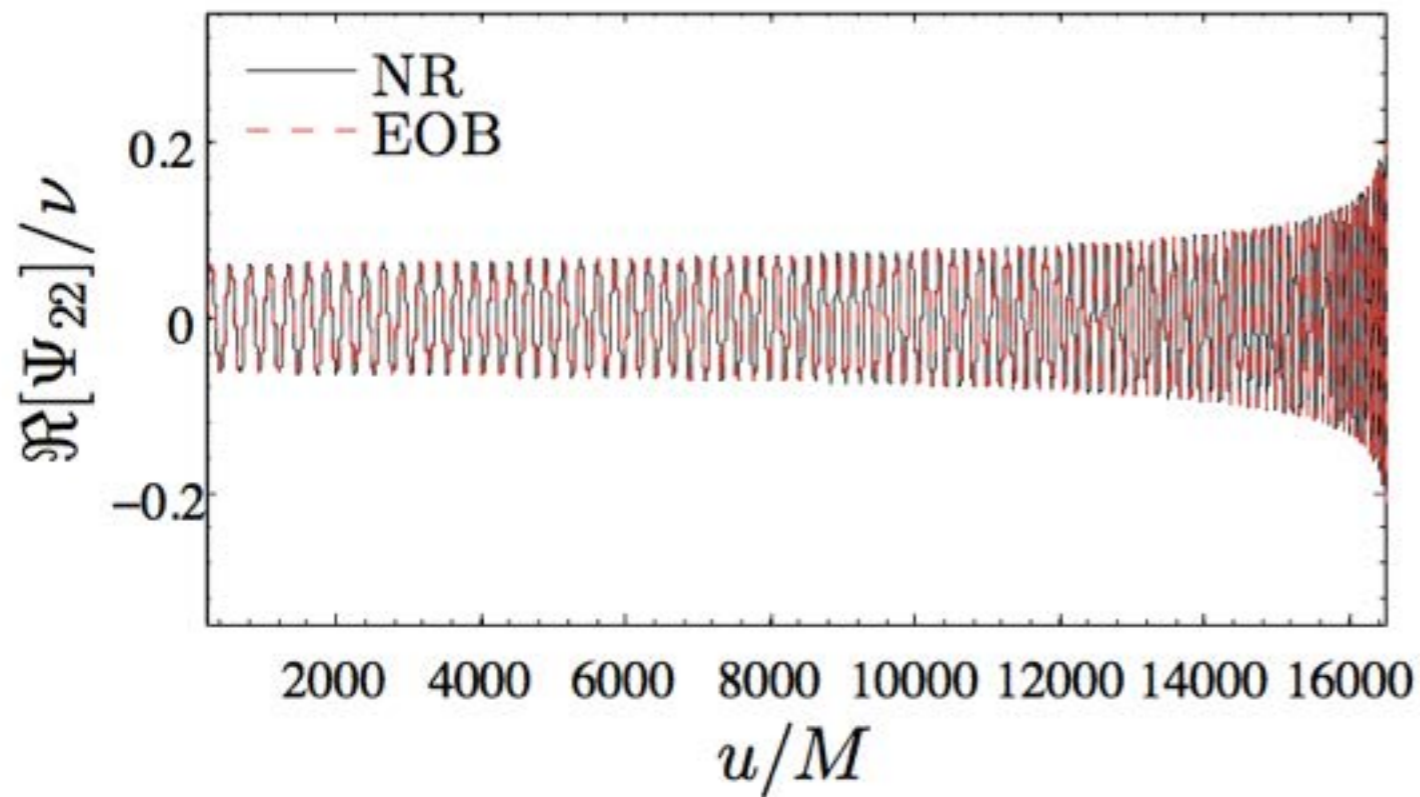
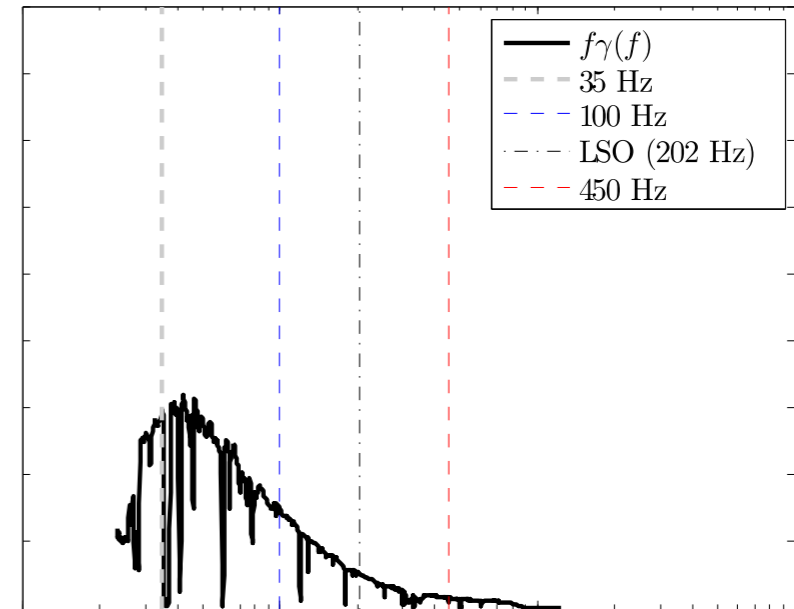
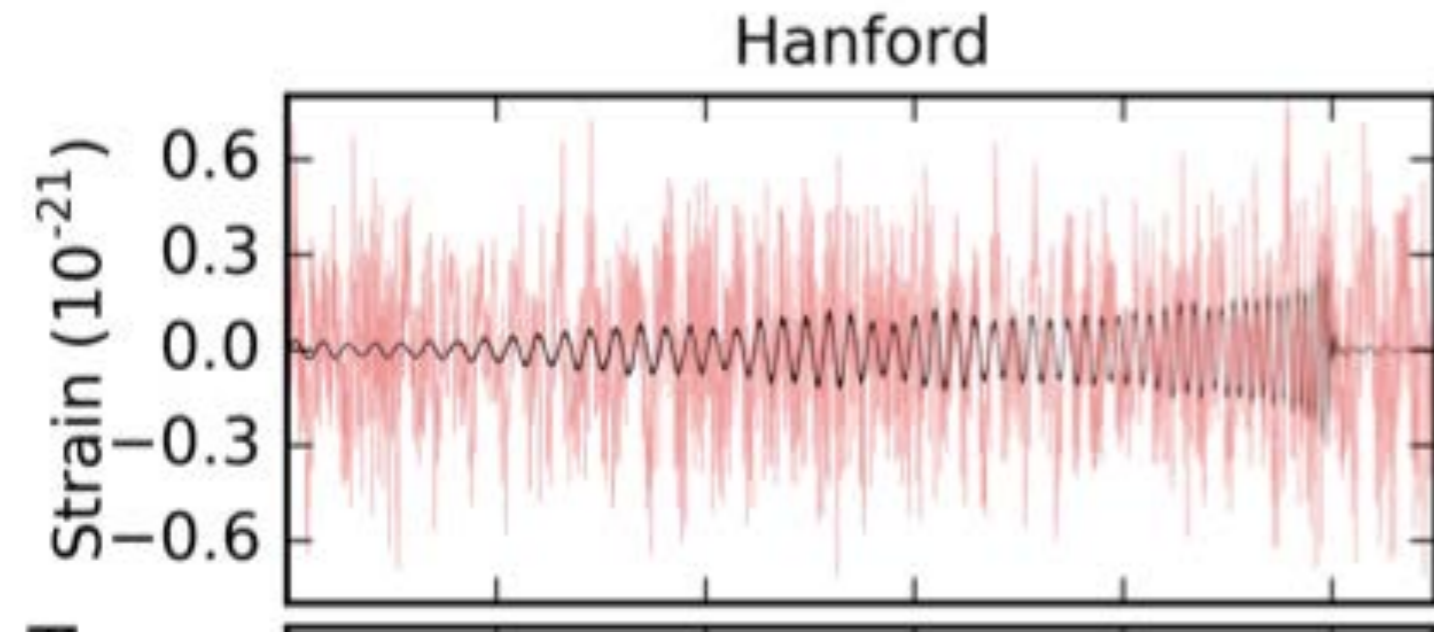
$$m_1 = 36_{-4}^{+5} M_{\odot}$$

$$m_2 = 29_{-4}^{+4} M_{\odot}$$

$$\chi_{\text{eff}} = -0.06_{-0.18}^{+0.17}$$

$$D_L = 410_{-180}^{+160} \text{Mpc}$$

GW151226: only detected via accurate matched filters



$$m_1 = 14.2_{-3.7}^{+8.3} M_{\odot}$$

$$m_2 = 7.5_{-2.3}^{+2.3} M_{\odot}$$

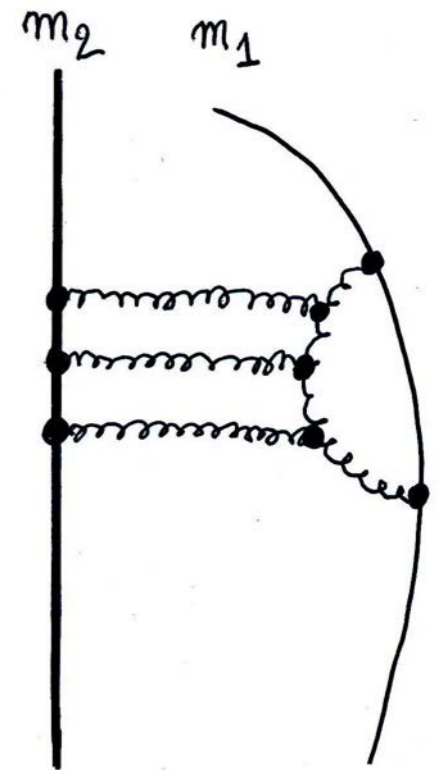
$$\chi_{\text{eff}} = +0.21_{-0.10}^{+0.20}$$

$$D_L = 440_{-190}^{+180} \text{Mpc}$$

FUTURE PROSPECT: GW DETECTORS IN SPACE LISA

Here also analytical methods (completed by numerical ones) will be important: **Gravitational Self-Force program: $m_1 \ll m_2$**

- Analytical high-PN results : Blanchet-Detweiler-LeTiec-Whiting '10, Damour '10, Blanchet et al '10, LeTiec et al '12, Bini-Damour '13-15, Kavanagh-Ottewill-Wardell '15
- (gauge-invariant) Numerical results : Detweiler '08, Barack-Sago '09, Blanchet-Detweiler-LeTiec-Whiting '10, Barack-Damour-Sago '10, Shah-Friedman-Keidl '12, Dolan et al '14, Nolan et al '15, ...
- Analytical PN results from high-precision (**hundreds to thousands** of digits !) numerical results : Shah-Friedman-Whiting '14, Johnson-McDaniel-Shah-Whiting '15



EOB[SF] program:
import high PN-SF results
in EOB (Bini-Damour '15,
Kavanagh et al '15,
Akçay, van de Meent,
Hopper,)

$$\begin{aligned}
 a_{10}^c &= \frac{18605478842060273}{7079830758000} \ln(2) - \frac{1619008}{405} \zeta(3) - \frac{21339873214728097}{1011404394000} \gamma \\
 &+ \frac{27101981341}{100663296} \pi^6 - \frac{6236861670873}{125565440} \ln(3) + \frac{360126}{49} \ln(2) \ln(3) + \frac{180063}{49} \ln(3)^2 \\
 &- \frac{121494974752}{9823275} \ln(2)^2 - \frac{24229836023352153}{549755813888} \pi^4 + \frac{1115369140625}{124540416} \ln(5) + \frac{96889010407}{277992000} \ln(7) \\
 &+ \frac{75437014370623318623299}{18690753201120000} - \frac{60648244288}{9823275} \ln(2) \gamma + \frac{200706848}{280665} \gamma^2 \\
 &+ \frac{11980569677139}{2306867200} \pi^2 + \frac{360126}{49} \gamma \ln(3), \\
 a_{10}^{\ln} &= -\frac{21275143333512097}{2022808788000} + \frac{200706848}{280665} \gamma - \frac{30324122144}{9823275} \ln(2) + \frac{180063}{49} \ln(3), \\
 a_{10}^{\ln^2} &= \frac{50176712}{280665},
 \end{aligned}$$

CONCLUSIONS

- Several aspects of Analytical Relativity have played a key role in the recent LIGO observations and data analysis: perturbative theory of motion, perturbative theory of GW generation, EOB formalism. The analytical EOB method had predicted in 2000 the complete GW signal emitted by the coalescence of two black holes. This was confirmed, and refined, in 2005 by Numerical Relativity. The union between Analytical Relativity and Numerical Relativity (particularly EOB+NR) has been crucial for computing the 250, 000 theoretical GW templates $h(t; m_1, m_2, S_1, S_2)$ which have been used for extracting the GW signals from the noise by matched filtering, for assessing their physical significance, and for measuring the source parameters. One expects most of the BBH (and BNS) signals to be detected only by means of accurate EOB[NR] templates (as was the case for GW151226).
- Quantum-related techniques and ideas (diagrammatic expansion of S_{eff} , Feynman integrals, dimensional regularization, quantum-inspired EOB correspondence, scattering amplitudes,) have been very useful. Many non-understood special features of the classical gravity of point-masses (effacing property beyond 5 loops, low-transcendentality of gravity diagrams,....) might benefit from further tapping the Quantum knowledge of the gravitational interaction.
- Opening of a new window on the universe: GW astronomy: might be dominated by BBH (Belczynski et al 2010); waiting for BNS + EM signal (GRB ?), and for LIGO/Virgo/Kagra/Indigo network. The detailed study of coalescences involving NS will open a window on the EOS of nuclear matter (tidal polarizability).
- New tests of strong-field GR and BH physics
- Window on cosmic-size strings especially via GW bursts above the Gaussian background
- Potentially new window on early universe via GW cosmological background
- Future GW detectors: space detectors, second generation ground-based detectors,...